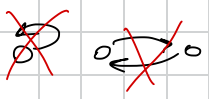
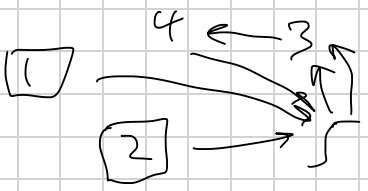
Braid variety cluster structures

joint w/ T. Lam, M. Sherman-Bennett, D. Speyer

Def. Quiver = digraph without 1- & 2-cycles, with vertices subdivided into frozen & mutable



Ex.



X - interesting variety $\rightsquigarrow$ coordinate ring $\mathbb{C}[X]$

Q - quiver on vertices $1,2,\dots,n$ $\rightsquigarrow$ cluster algebra $\mathcal{A}(Q) \subseteq \mathbb{C}(x_1, x_2, \dots, x_n)$.

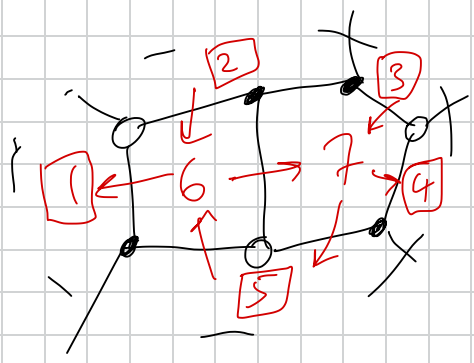
"Cluster structure on X": isomorphism $\mathbb{C}[X] \cong \mathcal{A}(Q)$.

Usually, Q & X - known/easy, proving $\mathbb{C}[X] \cong \mathcal{A}(Q)$ - hard.

Today, X - known, finding Q - hard.

Examples of cluster structures.

① X = positroid variety in $Gr(k,n)$. Q = planar dual of a plabic graph.



arrows:



Cluster structure: [G.-Lam '19].

② X = double Bruhat cell. [BFZ '99] $B_- vB_- \cap BwB \subset G$

$v, w \in S_n$. $\beta \in (\pm I)^{\ell(v)+\ell(w)}$ - double red. word.

$I = \{1,2,\dots,n-1\}$. $I, -I$ commute.

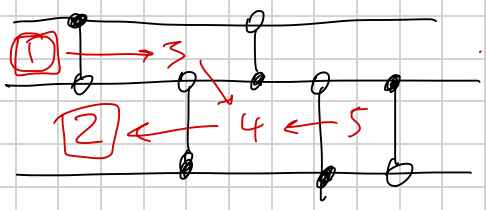
β = shuffle of (red. word for w on $+$ indices) and (red. word for v on $-$ indices).

Q = planar dual of plabic graph.

Ex. $v = s_2 s_1$ $w = s_1 s_2 s_1$ $n=3$ $s_i = (i \;\; i+1) \in S_n$

$\beta = -2 \;\; 1 \;\; 2 \;\; 1 \;\; -1$

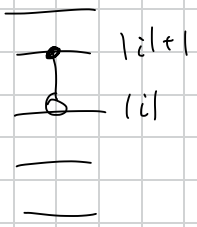
G_β:



$i \in I$:



$i \in -I$:



Goodearl-Yakimov '20

③ X = Braid variety. (generalizes both ① & ②)

$X = X_{u,\beta}$, where $u \in S_n$, $\beta \in (\pm I)^m$ - braid word such that $u \leq \delta(\beta)$

Demazure operators on S_n. $i \in I$, $u \in S_n$

$$s_i^{\uparrow}(u) = \begin{cases} s_i u, & \text{if } \ell(s_i u) > \ell(u) \\ u & \text{if } \ell(s_i u) < \ell(u) \end{cases} \qquad s_{-i}^{\uparrow}(u) = \begin{cases} u s_i, & \text{if } \ell(u s_i) > \ell(u) \\ u & \text{if } \ell(u s_i) < \ell(u) \end{cases}$$

$$s_i^{\downarrow}(u) = \begin{cases} u & > \\ s_i u & < \end{cases} \qquad s_{-i}^{\downarrow}(u) = \begin{cases} u & > \\ u s_i & < \end{cases}$$

Let $\beta = i_1 i_2 \cdots i_m \in (\pm I)^m$.

Let $u_m = u$ for $c = m, m-1, \ldots, 1$, $u_{c-1} := s_{i_c}^{\downarrow}(u_c)$.

Def: $u \le \delta(\beta) \iff \boxed{u_0 = id.}$

$1 \le c \le m$ — <u>solid</u> if $u_{c-1} = u_c$ ← vertices of $Q_{u,\beta}$.
hollow if $u_{c-1} \neq u_c$.

Let d-solid. Define $u_m^{<d>} = u_m, \ldots, u_d^{<d>} = u_d$

$u_{d-1}^{<d>} = s_{i_d}^{\uparrow}(u_d)$ — "defect"

$u_{c-1}^{<d>} = s_{i_c}^{\downarrow}(u_c^{<d>})$ for $c = d-1, \ldots, 1$.

d-mutable if $u_0^{<d>} = id$
frozen if $u_0^{<d>} \neq id$.

Grid minors factor as follows:

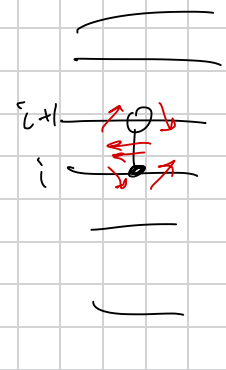
$i \in I$, $0 \le c \le m$, let $[i] = \{1, 2, \ldots, i\}$.

$\Delta_{c,i} = \prod_{d\text{-solid}:\ u_c^{<d>}([i]) \neq u_c([i])} x_d$ $\qquad$ $\Delta_{c,-i} = \prod_{d\text{-solid}:\ (u_c^{<d>})^{-1}([i]) \neq u_c^{-1}([i])} x_d$.

35

Quiver $Q_{u,\beta}$: sum up half-arrows:

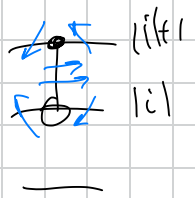
for each d-solid, if $i = i_d \in I$:



in $\pi_{red}(G_{u,\beta})$

[BFZ'95]

if $i = i_d \in -I$



in $\pi_{blue}(G_{u,\beta})$

Thm. [GLSBS]

$$\mathbb{C}[X] \cong \mathcal{A}(Q_{u,\beta}).$$

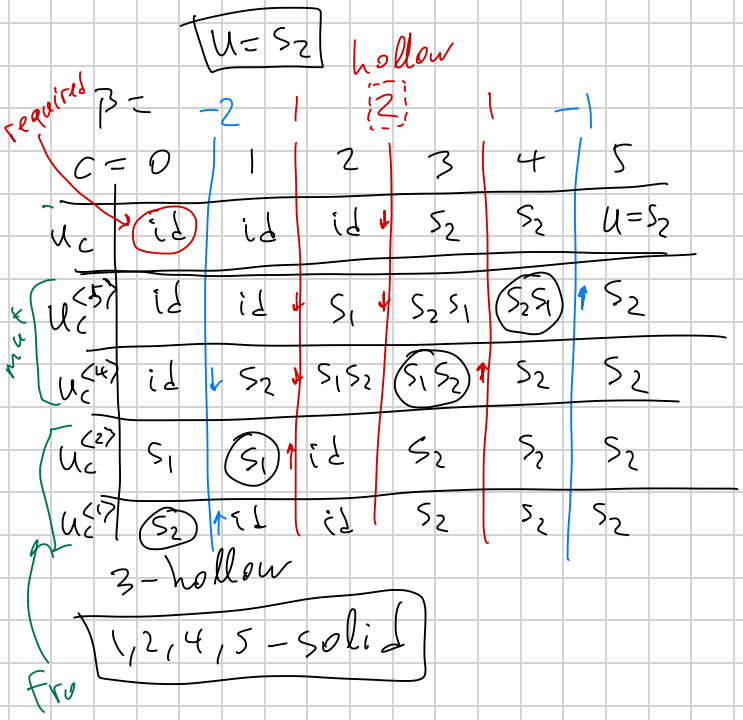
<u>Alt. constr. of $Q_{u,\beta}$:</u>

[Goncharov–Kenyon'13]: G-plabic graph → conjugate surface $S(G)$.

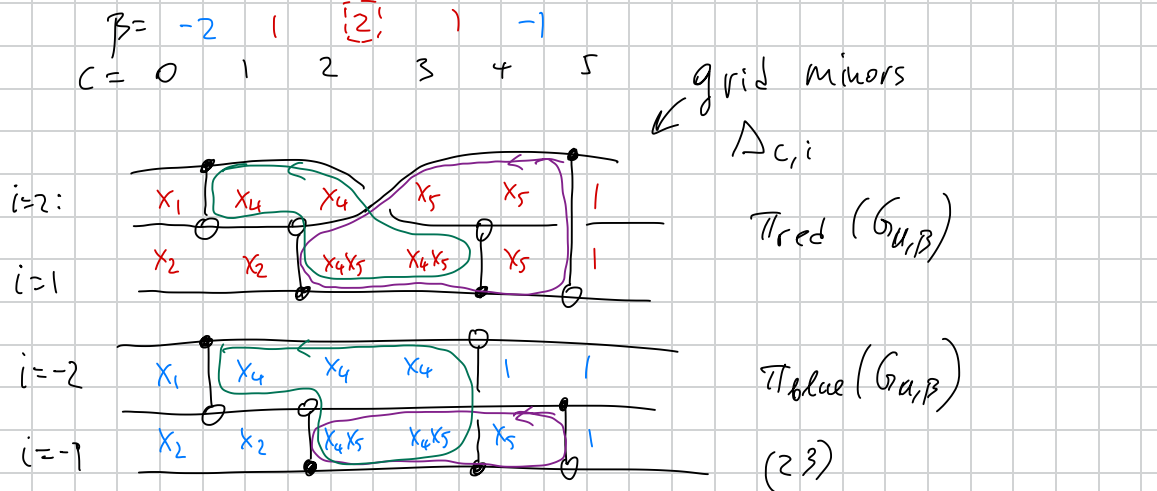
Faces of G → cycles in $S(G)$

arrows of Q = intersection form on $S(G)$





$Q_{u,\beta}$ = 1 5 2 4



$c=3$, $i=1$, $[i]=\{1\}$, $u_c = s_2$ $\quad u_c([i]) = \{1\}$.

$d=5$: $u_c^{<d>} = s_2 s_1$ $\quad s_2 s_1(1) = s_2(2) = 3$

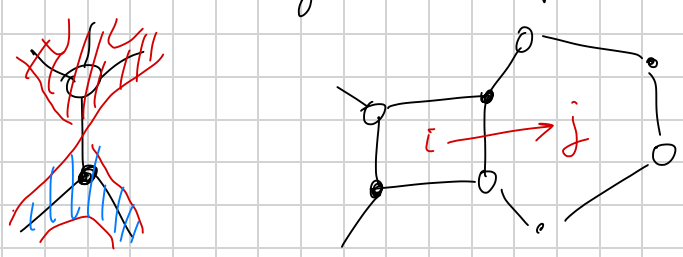
$u_c^{<d>}([i]) = \{3\} \neq \{1\} = u_c([i]) \Rightarrow x_5$

$d=4$: $u_c^{<d>} = s_1 s_2$ $\quad s_1 s_2(1) = s_1(1) = 2 \neq 1$

$\Rightarrow x_4$

Quiver $Q_{u,\beta}$.

[Goncharov-Kenyon '13] plabic graph $G \longrightarrow$ conjugate surface $S(G)$.



faces of $G \to$ cycle in $S(G)$

Arrows in $Q \leftarrow$ intersection form of $S(G)$

Thm [GLSBS '24] $\mathcal{A}(Q_{u,\beta}) \cong \mathbb{C}[X_{u,\beta}]$ see also [CGGLSS '24]

Constr. of $Q_{u,\beta}$:

Define 3D plabic graph $G_{u,\beta}$ – ribbon graph $\rightsquigarrow S(G_{u,\beta})$

d-solid $\rightsquigarrow$ cycle in $G_{u,\beta}$

$Q_{u,\beta}$ = intersection form on $S(G_{u,\beta})$.