Critical varieties in the Grassmannian

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University of Michigan Combinatorics Seminar
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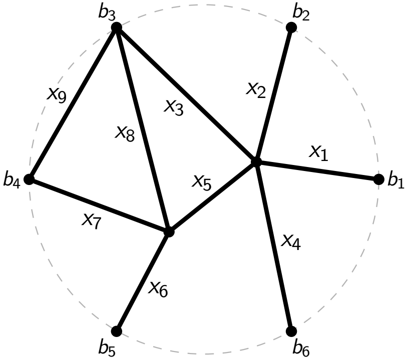
Motivation: Ising model

- $(G, \mathbf{x})$: weighted graph embedded in a disk.



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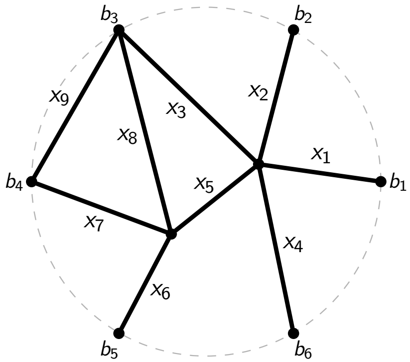

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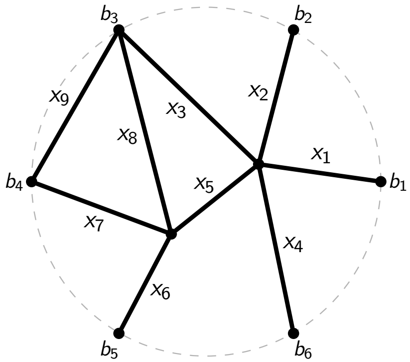

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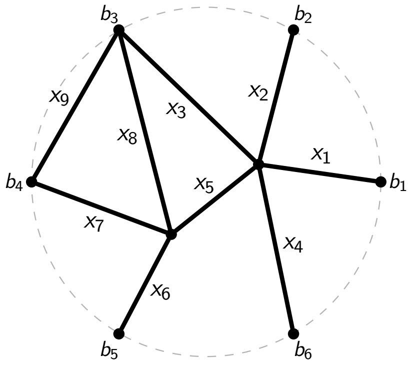
$$\text{Prob}(\sigma) := \frac{1}{Z} \prod_{\substack{\{u,v\} \in E(G): \\ \sigma_u \neq \sigma_v}} x_{\{u,v\}},$$

where Z is such that the total probability is 1.



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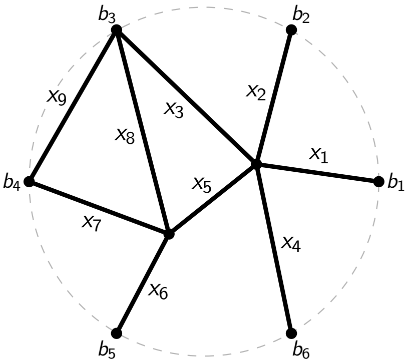
- Boundary correlation:
  $\langle \sigma_i \sigma_j \rangle := \mathrm{Prob}(\sigma_{b_i} = \sigma_{b_j}) - \mathrm{Prob}(\sigma_{b_i} \neq \sigma_{b_j})$.



Phase transition

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Usually:

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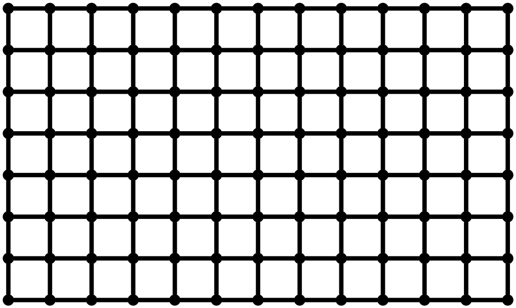

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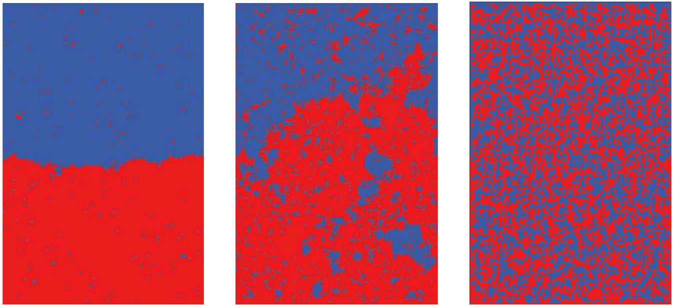

$x < x_{\mathrm{crit}}$ $x = x_{\mathrm{crit}}$ $x > x_{\mathrm{crit}}$

Picture credit: Dmitry Chelkak

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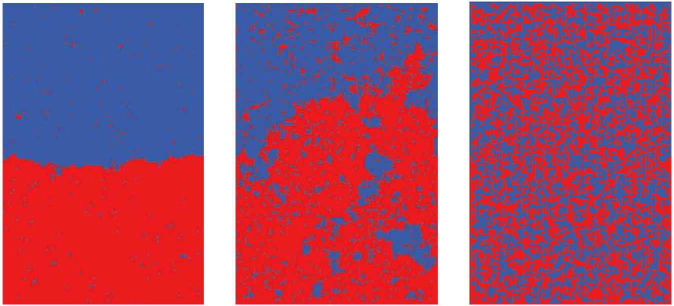
- Get a phase transition at critical temperature x_{crit}.
- Square lattice: $x_{\text{crit}} = \sqrt{2} - 1$.



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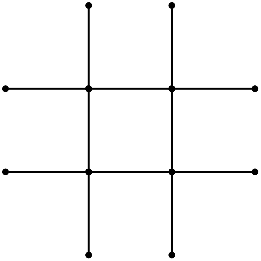

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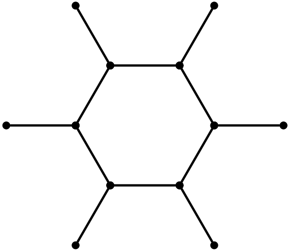
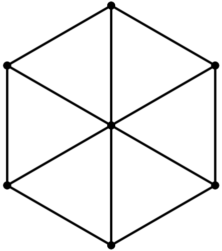


Hexagonal lattice
$x_{\text{crit}} = 2 - \sqrt{3}$



Triangular lattice
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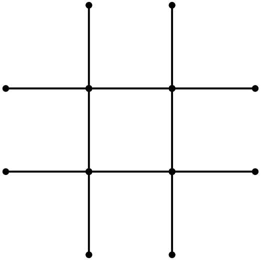
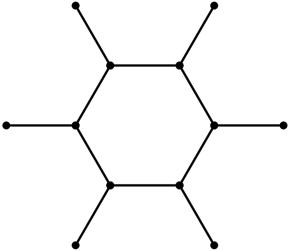


Square lattice

$x_{\text{crit}} = \sqrt{2} - 1$

$x_{\text{crit}} = \tan(\pi/8)$



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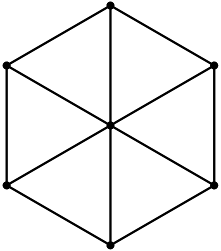

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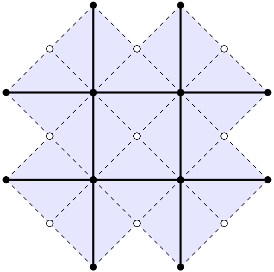

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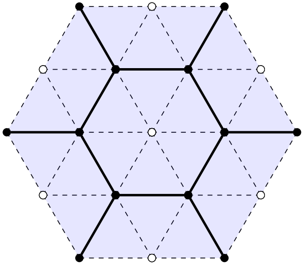

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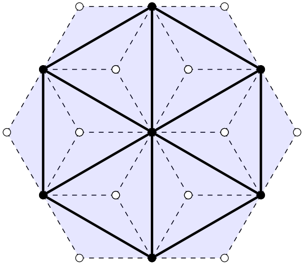

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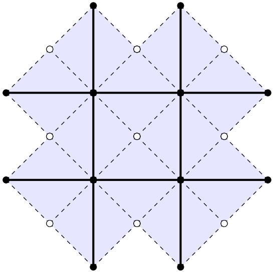

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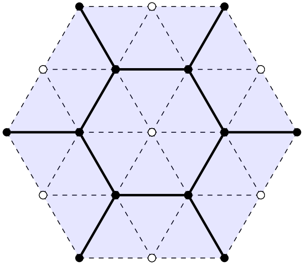

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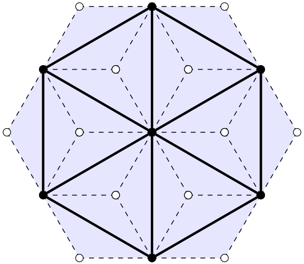

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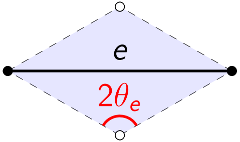

$$x_e = \tan(\theta_e/2)$$

Critical Z-invariant Ising model

[Bax86] R. J. Baxter. Free-fermion, checkerboard and **Z**-invariant lattice models in statistical mechanics. *Proc. Roy. Soc. London Ser. A*, 404(1826):1–33, 1986.

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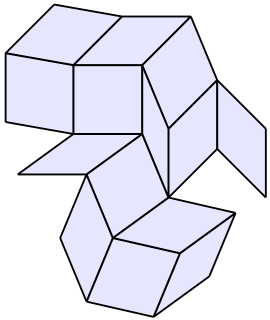

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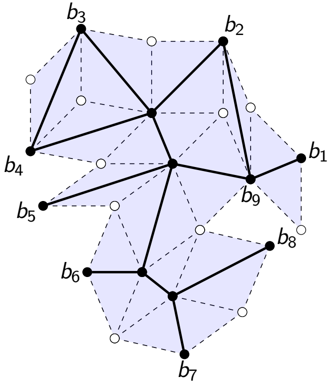

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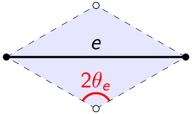

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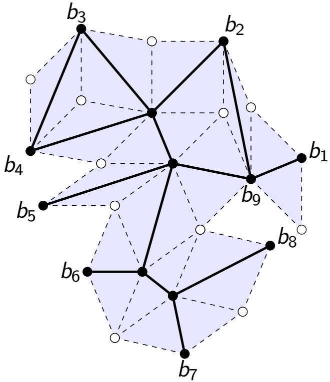

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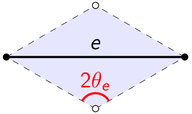

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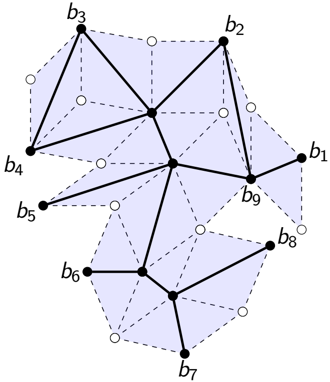

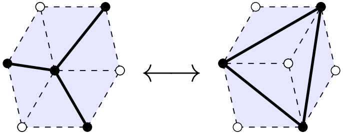

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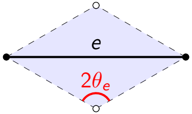

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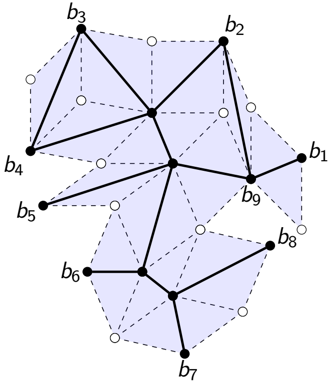
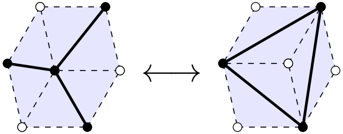
- Conclusion: $\langle \sigma_i \sigma_j \rangle_R$ depends only on the polygonal region R.





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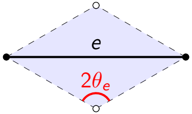

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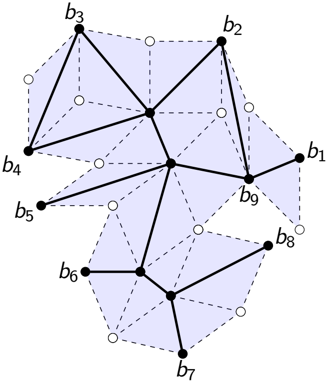
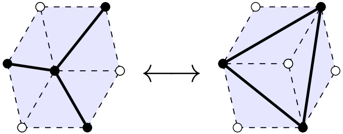
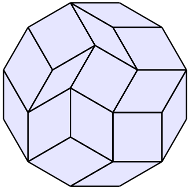
- Formula for $\langle \sigma_i \sigma_j \rangle_R$ in terms of R?

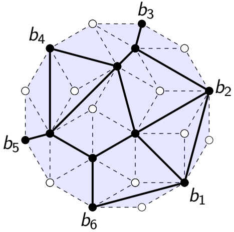




A formula for regular polygons

Let R_N be a regular $2N$-gon and $\langle \sigma_i \sigma_j \rangle_{R_N}$ be the corresponding boundary correlations.





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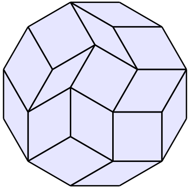

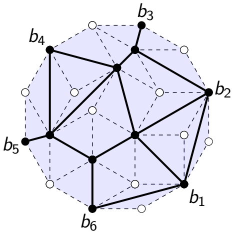

Theorem (G. (2020))

For $1 \leqslant i, j \leqslant N$ and $d := |i - j|$, we have

$$\langle \sigma_i \sigma_j \rangle_{R_N} = \frac{2}{N} \left(\frac{1}{\sin((2d-1)\pi/2N)} - \frac{1}{\sin((2d-3)\pi/2N)} + \cdots \pm \frac{1}{\sin(\pi/2N)} \right) \mp 1.$$

[Gal20] Pavel Galashin. A formula for boundary correlations of the critical Ising model. arXiv:2010.13345, 2020.

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- Q: Does $\langle \sigma_1 \sigma_{d+1} \rangle_{R_N} \to 0$ for $1 \ll d \ll N$?
- A: Yes, by the Leibniz formula for π:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots .$$

Theorem (G. (2020))

If R_N is a regular $2N$-gon then for $1 \leqslant i, j \leqslant N$ and $d := |i-j|$, we have

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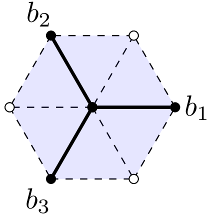

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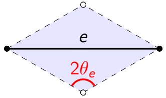

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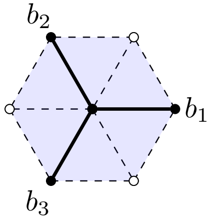

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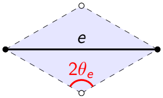

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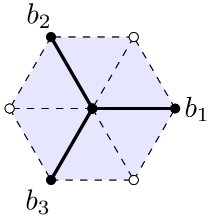

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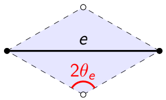
- Resistance R_e = ratio of diagonals: 

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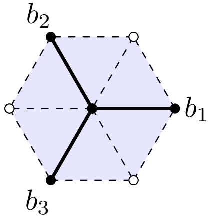

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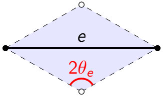

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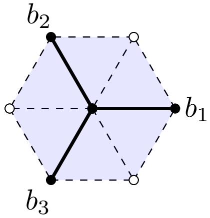

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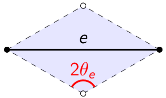

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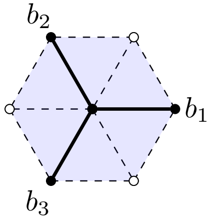

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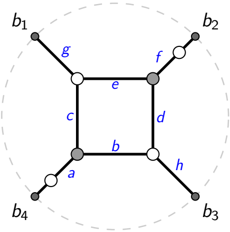
Critical varieties

- (G, wt) – a weighted planar bipartite graph, with n black boundary vertices $b_1, b_2, \dots, b_n$ of degree 1.



[Pos06] Alexander Postnikov. Total positivity, Grassmannians, and networks. Preprint, `http:// math.mit.edu/~apost/papers/tpgrass.pdf`, 2006.

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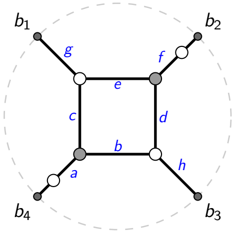

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- Boundary measurement map $\mathrm{Meas}_G(\mathrm{wt}) = (\Delta_J(G, \mathrm{wt}))_{J \in \binom{[n]}{k}}$:

$$\Delta_J(G, \mathrm{wt}) := \sum_{\mathcal{A}: \partial(\mathcal{A}) = J} \mathrm{wt}(\mathcal{A}), \quad \text{where} \quad \mathrm{wt}(\mathcal{A}) := \prod_{e \in \mathcal{A}} \mathrm{wt}(e).$$

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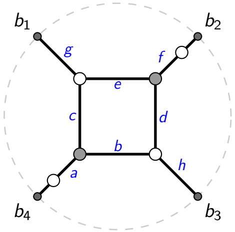

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- A strand is a path in G that makes a sharp right turn at each black vertex and a sharp left turn at each white vertex.
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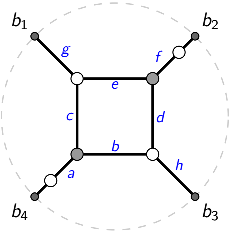

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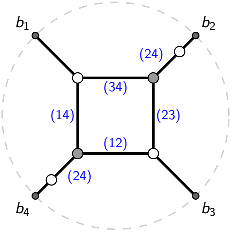

$$(pq) := \sin(\theta_q - \theta_p)$$

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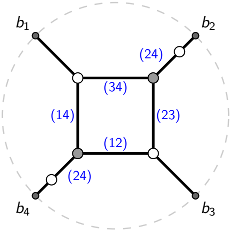

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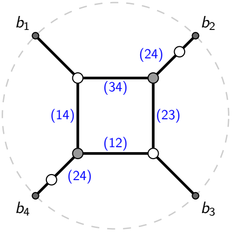

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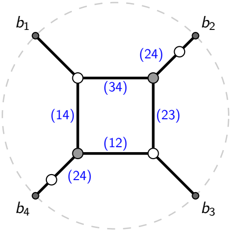

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\Delta_{12} &= (23) \cdot (24) \\
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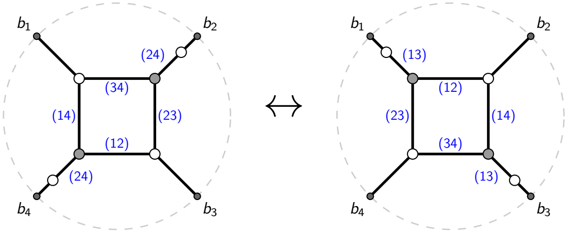
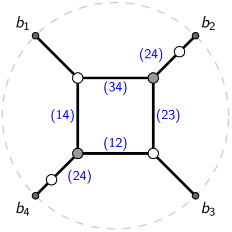
- This model is invariant under square moves:





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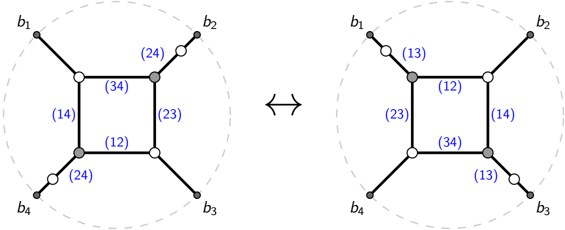
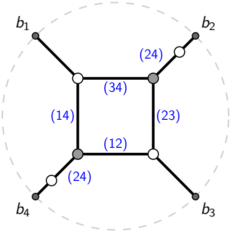


- A graph G is reduced if it has the minimal number of faces among all graphs with the same strand permutation.



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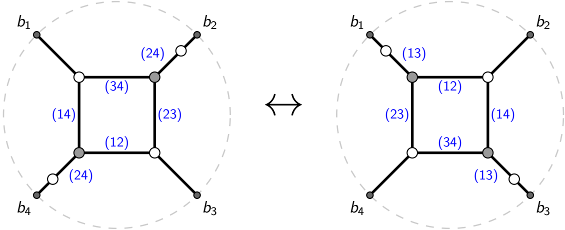

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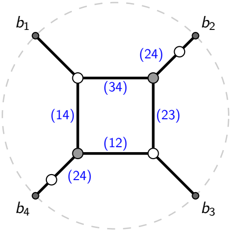

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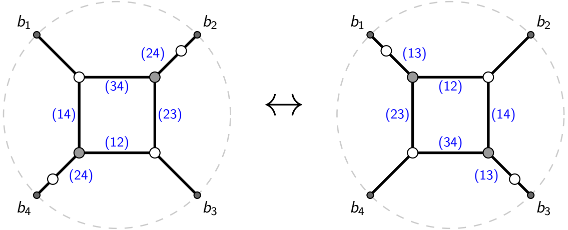

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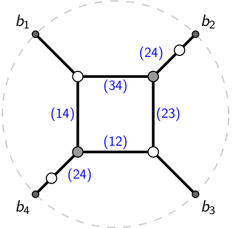
- Conclusion: for each reduced G with $f_G = f$, $\mathrm{Meas}_G(\mathrm{wt}_{\boldsymbol{\theta}}) = \mathrm{Meas}_f(\boldsymbol{\theta})$ depends only on f and $\boldsymbol{\theta}$.



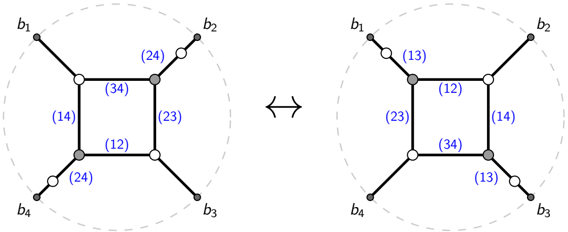
$$(pq) := \sin(\theta_q - \theta_p)$$

$\Delta_{12} = (23) \cdot (24)$
$\Delta_{23} = (34) \cdot (24)$
$\Delta_{34} = (14) \cdot (24)$
$\Delta_{14} = (12) \cdot (24)$
$\Delta_{13} = (24) \cdot (24)$
$\Delta_{24} = (14) \cdot (23) + (12) \cdot (34) = (13) \cdot (24)$

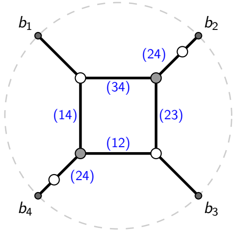
by Ptolemy's theorem

Critical dimer model

- This model is invariant under square moves:



- A graph G is reduced if it has the minimal number of faces among all graphs with the same strand permutation.
- Any two reduced G, G' with $f_G = f_{G'}$ are related by square moves.
- Conclusion: for each reduced G with $f_G = f$, $\mathrm{Meas}_G(\mathrm{wt}_{\boldsymbol{\theta}}) = \mathrm{Meas}_f(\boldsymbol{\theta})$ depends only on f and $\boldsymbol{\theta}$.
- Formula for $\mathrm{Meas}_f(\boldsymbol{\theta})$ in terms of f and $\boldsymbol{\theta}$?



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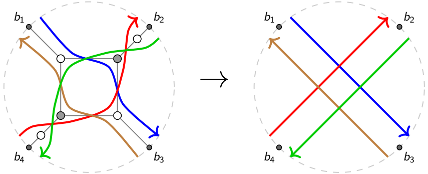
\Delta_{24} &= (14)\cdot(23)+(12)\cdot(34) = (13)\cdot(24)
\end{aligned}$$

- Recall: $f \in S_n$. The reduced strand diagram of f is obtained by drawing an arrow $b_s^+ \to b_p^-$ if $f(s) = p$.



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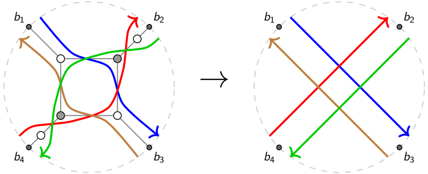

$J_1 = \{2\}$ $J_2 = \{3\}$ $J_3 = \{4\}$ $J_4 = \{1\}$

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- Consider a curve $\gamma_{f,\theta}(t) = (\gamma_1(t), \gamma_2(t), \ldots, \gamma_n(t))$:

$$\gamma_r(t) := \epsilon_r \prod_{p \in J_r} \sin(t - \theta_p) \qquad \text{for } r \in [n],$$

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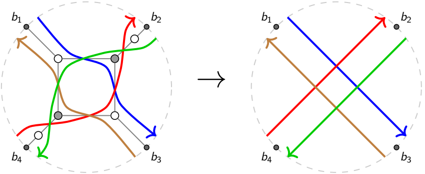

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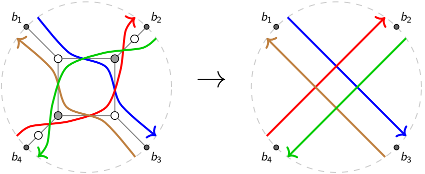
$$\mathrm{Meas}_f(\boldsymbol{\theta}) = \mathrm{Span}(\boldsymbol{\gamma}_{f,\boldsymbol{\theta}}) \quad \textit{inside } \mathrm{Gr}(k,n).$$



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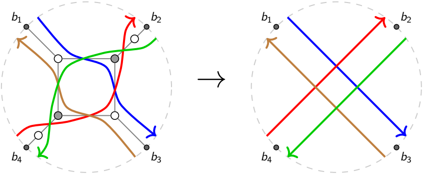

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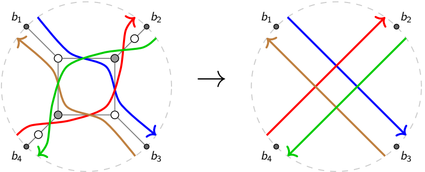

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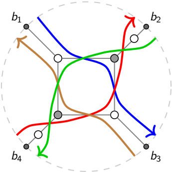

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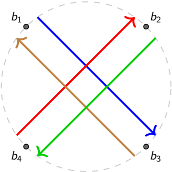

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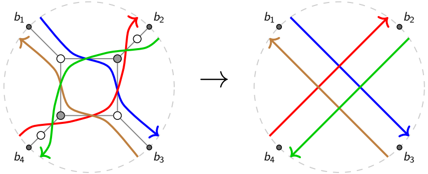

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$\mathrm{Span}(\boldsymbol{\gamma}_{f,\boldsymbol{\theta}})$ is the row span of

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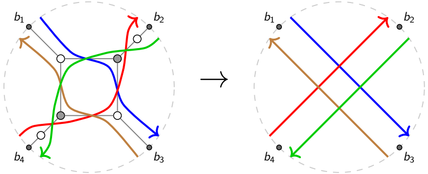

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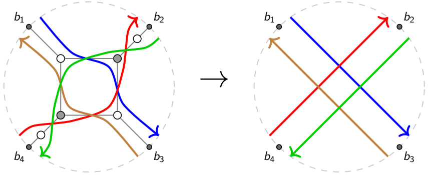

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[Pos06] Alexander Postnikov. Total positivity, Grassmannians, and networks. Preprint, `http:// math.mit.edu/~apost/papers/tpgrass.pdf`, 2006.

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- **Goal**: for each $f \in S_n$, introduce

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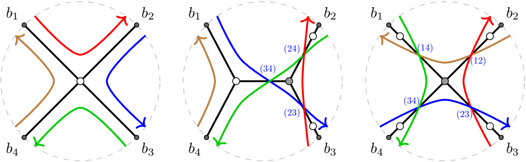

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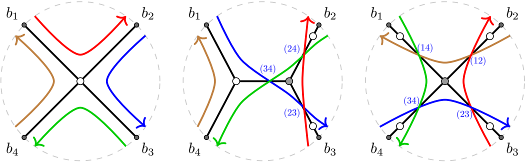

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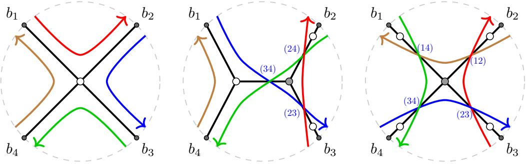

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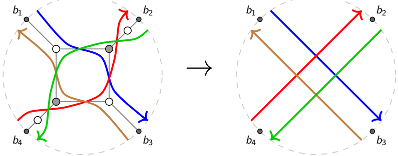

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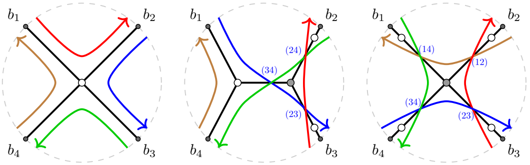

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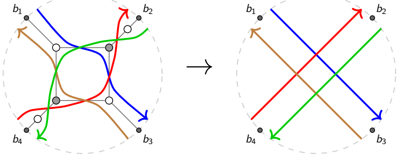

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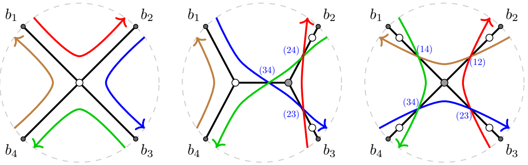

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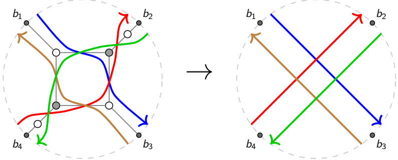

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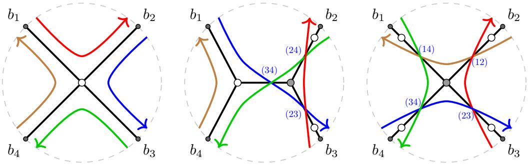

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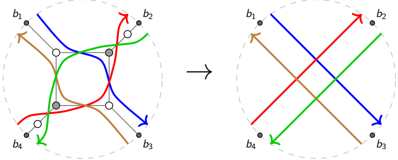

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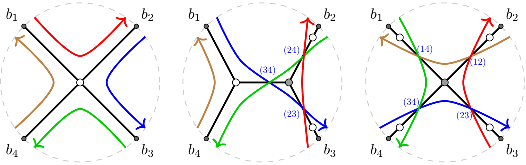

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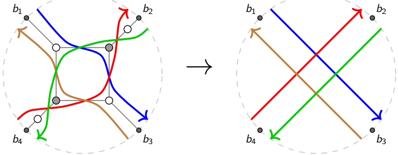

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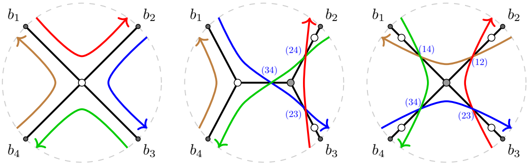

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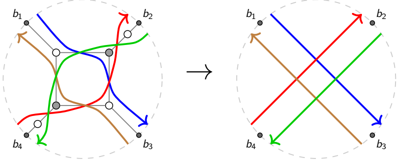

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Theorem

This holds for the top cell.

(top cell: $f_{k,n} \in S_n$ sending $p \mapsto p + k \bmod n$ for all p)

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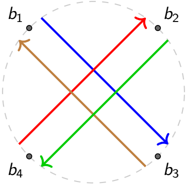

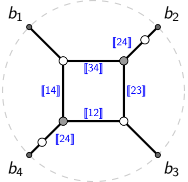

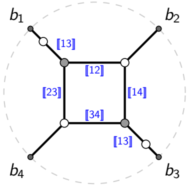

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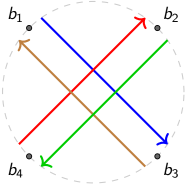

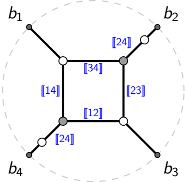

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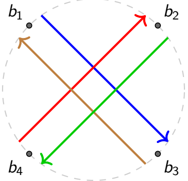

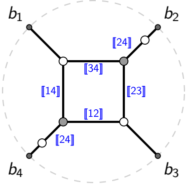

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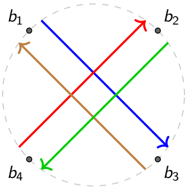

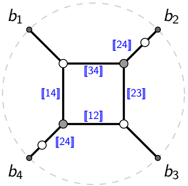

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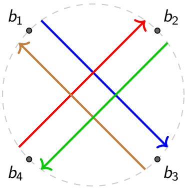

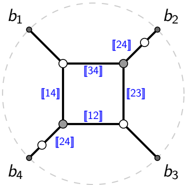

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"Ptolemy's theorem over $\mathbb{C}$"

$$[\![pq]\!] := [\![t_q, t_p]\!]$$

- Make a change of variables: $t_p := \exp(i\theta_p)$.
- $\sin(\theta_q - \theta_p) = [\![t_q, t_p]\!] := \dfrac{1}{2i}(t_q/t_p - t_p/t_q)$.

Definition

$\mathbf{t} \in (\mathbb{C}^*)^n$ is called f-admissible if for all $p < q$ such that the arrows $b_s^+ \to b_p^-$ and $b_t^+ \to b_q^-$ cross, we have $t_p \neq \pm t_q$.

Theorem

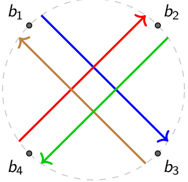
$\mathbf{t}$ is f-admissible $\Longrightarrow$ there is a well-defined element

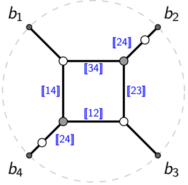
$$\mathrm{Meas}_f(\mathbf{t}) \in \Pi_f^\circ.$$

- Reason: Laurent phenomenon for critical varieties.
- Define the open critical variety

$$\mathrm{Crit}_f^\circ := \{\mathrm{Meas}_f(\mathbf{t}) \mid \mathbf{t} \in (\mathbb{C}^*)^n \text{ is } f\text{-admissible}\}.$$

- Critical variety Crit_f = Zariski closure of $\mathrm{Crit}_f^{>0}$ (or of Crit_f°).
- Problem: show that Crit_f° is an open subvariety of Crit_f and describe them by polynomial equations.





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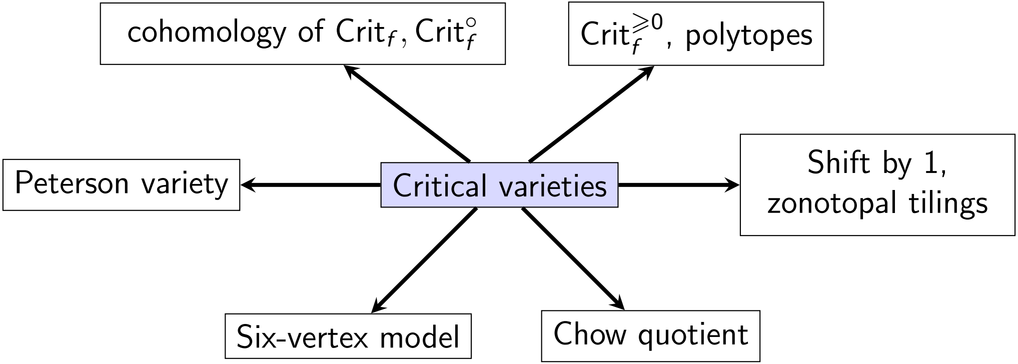
- If $\theta_r = r\pi/n$ for all $1 \leqslant r \leqslant n$, we get $\mathrm{Meas}_{f_{k,n}}(\boldsymbol{\theta}) = X_0^{(k,n)}$.
- This yields the above formulas for regular polygons in the Ising and electrical cases.

Thanks!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