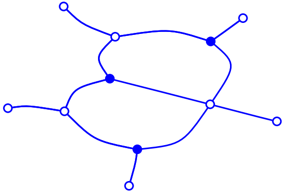
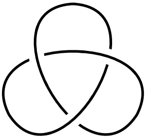
Positroids, knots, and q, t-Catalan numbers

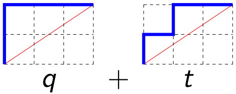
Pavel Galashin (UCLA)

Cornell University Oliver Club
February 9, 2023

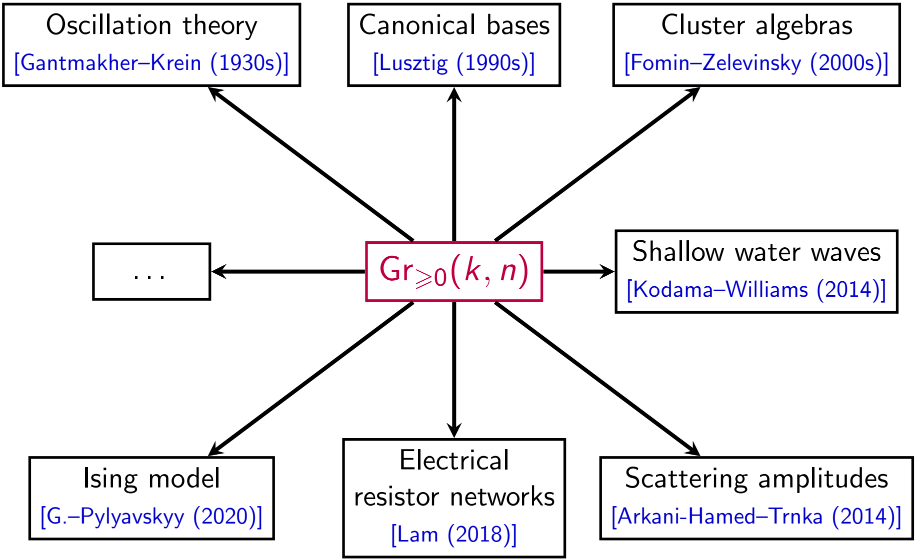
Joint work with Thomas Lam







Motivation: total positivity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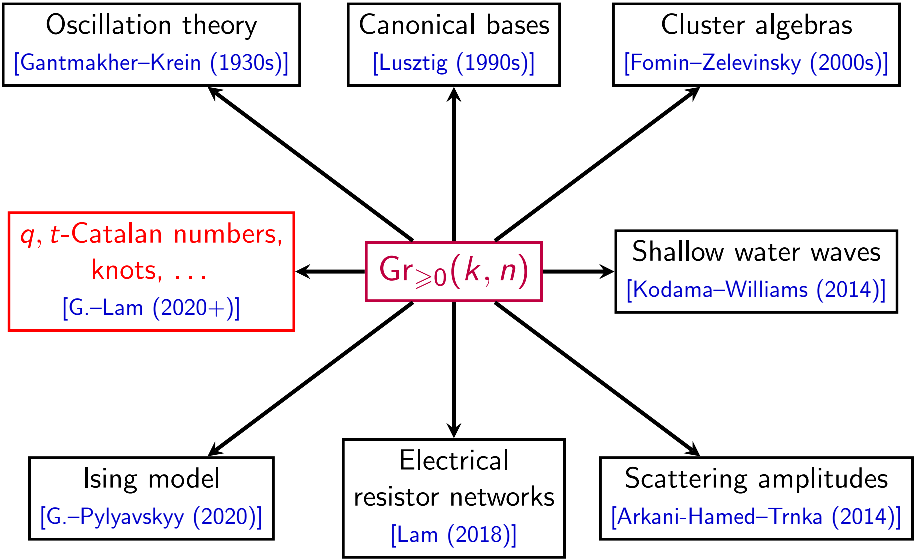
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This is a horrible stratification. [Mnëv (1988)]

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[Pos06] A. Postnikov. Total positivity, Grassmannians, and networks. arXiv:math/0609764.

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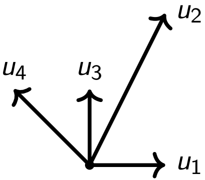

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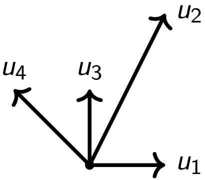

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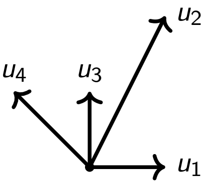

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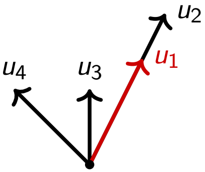

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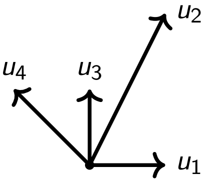

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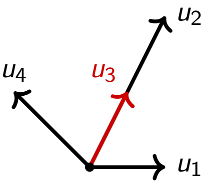

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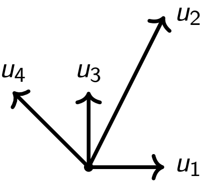

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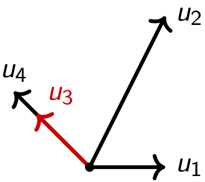

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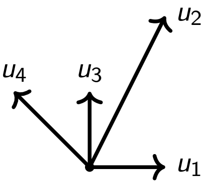

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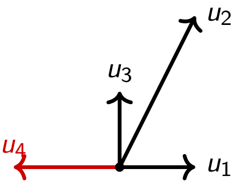

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Top cell: $\Delta_{13}, \Delta_{24}, \Delta_{12}, \Delta_{34}, \Delta_{14}, \Delta_{23} > 0$.
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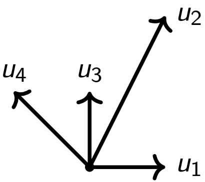
Example: $\mathrm{Gr}_{\geqslant 0}(2,4)$

$$\mathrm{RowSpan}\begin{array}{c}\begin{bmatrix}1 & 1 & 0 & -1\\ 0 & 2 & 1 & 1\end{bmatrix}\\ \begin{matrix}u_1 & u_2 & u_3 & u_4\end{matrix}\end{array} \in \mathrm{Gr}_{\geqslant 0}(2,4)$$



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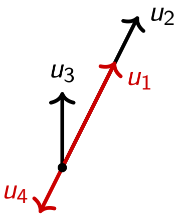

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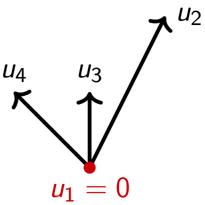

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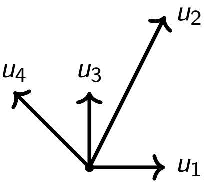

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Question

What is the topology of $\mathrm{Gr}_{\geqslant 0}(k,n)$ and of its boundary cells?

Topology of $\mathrm{Gr}_{\geqslant 0}(k, n)$

Theorem (Postnikov (2006))

Each boundary cell (some $\Delta_I > 0$ and the rest $\Delta_J = 0$) is an open ball.

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Theorem (G.–Karp–Lam)

[GKL1] $\mathrm{Gr}_{\geqslant 0}(k,n)$ *is homeomorphic to a closed ball.*

[GKL2] *The closure of each cell is homeomorphic to a ball.*

[GKL1] P. Galashin, S. Karp, and T. Lam. The totally nonnegative Grassmannian is a ball. *Adv. Math.*, 397: 108123, 2022.

[GKL2] P. Galashin, S. Karp, and T. Lam. Regularity theorem for totally nonnegative flag varieties. *J. Amer. Math. Soc.*, 35(2):513–579, 2021.

Combinatorics of positroids

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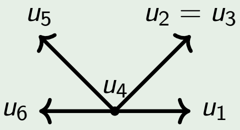
Example

$$\begin{array}{c} \begin{bmatrix} 1 & 1 & 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix} \\ \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \end{array} \quad \begin{array}{c} \begin{bmatrix} 1 & 1 & 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix} \\ \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \end{array}$$



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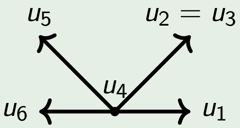

$$f_V = \begin{pmatrix} 1 & & & & & \\ 5 & & & & & \end{pmatrix}.$$

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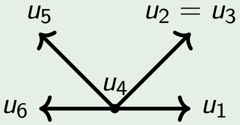
Example

$$\begin{array}{cccccc} \multicolumn{6}{c}{\begin{bmatrix} 1 & 1 & 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}} \\ u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \quad \begin{array}{cccccc} \multicolumn{6}{c}{\begin{bmatrix} 1 & 1 & 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}} \\ u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array}$$



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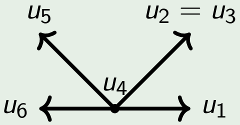

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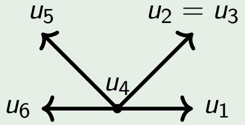

$$f_V = \begin{pmatrix} 1 & 2 & 3 & 4 & & \\ 5 & 3 & 6 & 4 & & \end{pmatrix}.$$

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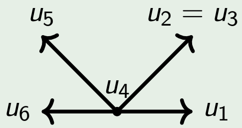

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\begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} & \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array}
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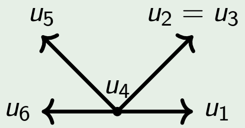

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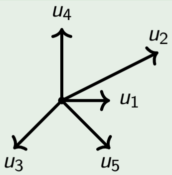

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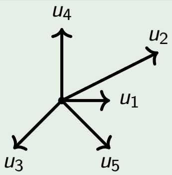
$$u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5 \qquad u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5$$



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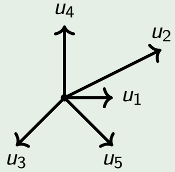

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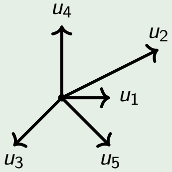

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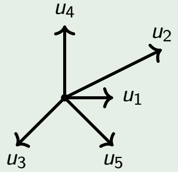

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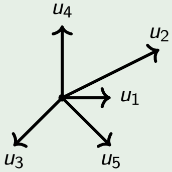

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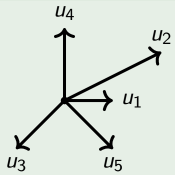

Let $f_{k,n}(i) \equiv i + k \pmod{n}$ for all $i = 1, 2, \ldots, n$. Then

$$f_V = f_{k,n} \quad \Longleftrightarrow \quad \Delta_{1,\ldots,k}, \Delta_{2,\ldots,k+1} \ldots, \Delta_{n,1,\ldots,k-1} \neq 0.$$

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- Both are defined for $\mathrm{Gr}(k,n;\mathbb{F})$ over arbitrary field $\mathbb{F}$.

Summary so far

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Question

What can we say about Π_f° over other fields $\mathbb{F}$?

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Given a variety [of "Hodge–Tate type"]

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Most interesting case: $\mathcal{P}(X; q, t) \in \mathbb{N}[q, t]$ (i.e., odd cohomology vanishes).

Given a variety [of "Hodge–Tate type"]

$$X(\mathbb{F}) = \{\mathbf{x} \in \mathbb{F}^r \mid P_1(\mathbf{x}) = \cdots = P_n(\mathbf{x}) = 0, \quad Q_1(\mathbf{x}) \neq 0, \ldots, Q_m(\mathbf{x}) \neq 0\}.$$

Step 1. Compute point count $\#X(\mathbb{F}_q)$ over $\mathbb{F}_q$.

Step 2. Compute Poincaré polynomial $\mathcal{P}(X(\mathbb{C}); t) := \sum_i t^{\frac{i}{2}} \dim H^i(X(\mathbb{C}))$.

Step 3. Compute the mixed Hodge polynomial $\mathcal{P}(X; q, t)$

[Deligne splitting / weight filtration $\to$ canonical second grading on $H^*(X)$]

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Question: Which variety should we choose?

$$\mathrm{Gr}(k,n;\mathbb{F}) := \{W \subseteq \mathbb{F}^n \mid \dim(W) = k\} = \frac{\{k \times n \text{ matrices of rank } k\}}{(\text{row operations})}.$$

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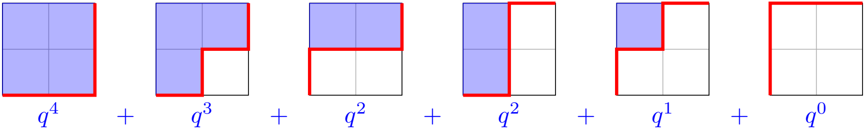

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- Reason: Schubert decomposition.

- Rational Catalan numbers: for $1 \leqslant k \leqslant n$ such that $\gcd(k, n) = 1$, let

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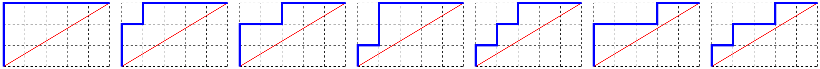

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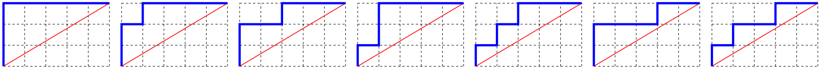

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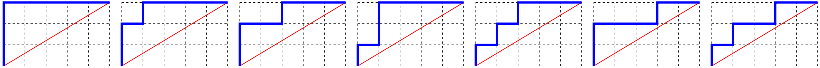

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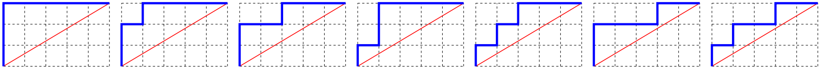

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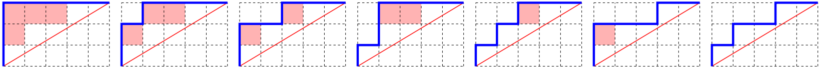

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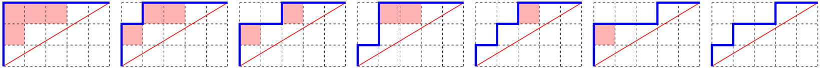

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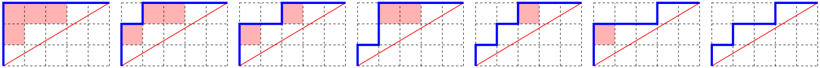

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Definition (G.–Lam)

Let $\gcd(k, n) = 1$. The Catalan variety is given by

$$X^\circ_{k,n} := \{V \in \mathrm{Gr}(k, n) \mid \Delta_{1,\dots,k}(V) = \Delta_{2,\dots,k+1}(V) = \cdots = \Delta_{n,1,\dots,k-1}(V) = 1\}.$$

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Example:

$$X^\circ_{2,5} = \left\{ \mathrm{RowSpan} \begin{pmatrix} 1 & 0 & a & b & c \\ 0 & 1 & d & e & f \end{pmatrix} \middle| \begin{array}{cc} -a = 1, & ae - bd = 1, \\ f = 1, & bf - ce = 1 \end{array} \right\}.$$

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$$\#X_{2,5}^{\circ}(\mathbb{F}_q) = q^2+1, \quad \mathcal{P}(X_{2,5}^{\circ}(\mathbb{C});t) = 1+t, \quad \mathcal{P}(X_{2,5}^{\circ};q,t) = q+t.$$

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Definition

Let $\gcd(k,n)=1$. Recall:
$\Pi^{\circ}_{k,n} := \{V \in \mathrm{Gr}(k,n) \mid \Delta_{1,\dots,k}(V), \Delta_{2,\dots,k+1}(V), \cdots, \Delta_{n,1,\dots,k-1}(V) \neq 0\}$.

Example:

$$\Pi^{\circ}_{2,5} = \left\{ \mathrm{RowSpan} \begin{pmatrix} 1 & 0 & a & b & c \\ 0 & 1 & d & e & f \end{pmatrix} \;\middle|\; \begin{matrix} -a \neq 0, & ae - bd \neq 0, \\ f \neq 0, & bf - ce \neq 0 \end{matrix} \right\}.$$

Theorem (G.–Lam)

$$\begin{tikzcd}[column sep=tiny]
 & \boxed{\mathcal{P}(\Pi^{\circ}_{k,n}; q,t) = (q^{\frac12} + t^{\frac12})^{n-1} C_{k,n-k}(q,t)} \arrow[dl, "t^{\frac12} = -q^{-\frac12}"'] \arrow[dr, "q^{\frac12} = 1"] & \\
\boxed{\#\Pi^{\circ}_{k,n}(\mathbb{F}_q) = (q-1)^{n-1} C'_{k,n-k}(q)} \arrow[dr, "q = 1"'] & & \boxed{\mathcal{P}(\Pi^{\circ}_{k,n}(\mathbb{C}); t) = (1 + t^{\frac12})^{n-1} C''_{k,n-k}(t)} \arrow[dl, "t^{\frac12} = -1"] \\
 & \boxed{\textit{Euler characteristic} = 0} &
\end{tikzcd}$$

Summary so far

- $V \in \mathrm{Gr}(k, n; \mathbb{F}) \longrightarrow$ Permutation $f_V : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$.
- Positroid stratification:

$$\mathrm{Gr}(k, n; \mathbb{F}) = \bigsqcup_f \Pi_f^\circ, \quad \text{where} \quad \Pi_f^\circ := \{V \in \mathrm{Gr}(k, n; \mathbb{F}) \mid f_V = f\}.$$

- Top-dimensional piece: $f_{k,n}(i) \equiv i + k \pmod n$,

$$\begin{aligned} \Pi_{k,n}^\circ &= \{V \in \mathrm{Gr}(k, n; \mathbb{F}) \mid f_V = f_{k,n}\} \\ &= \{\Delta_{1,\dots,k}, \Delta_{2,\dots,k+1} \dots, \Delta_{n,1,\dots,k-1} \neq 0\} \end{aligned}$$

Question

What can we say about Π_f° over other fields $\mathbb{F}$?

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What about arbitrary Π_f°?

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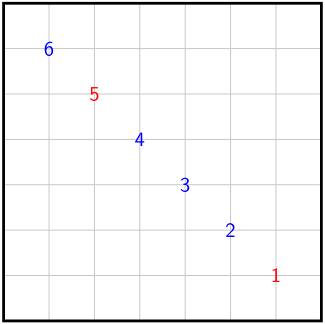

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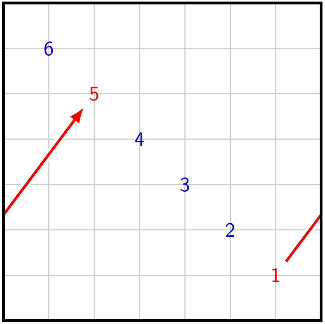

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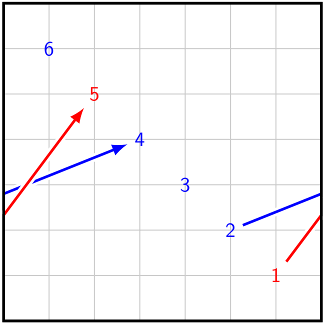

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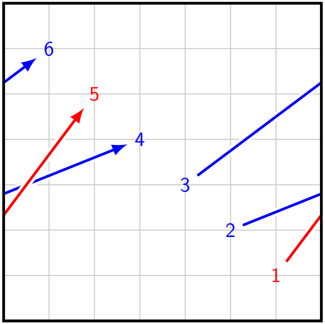

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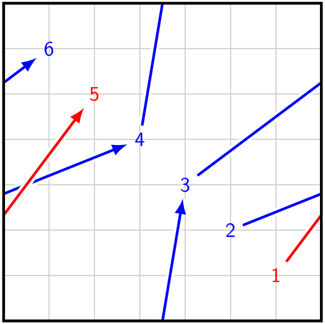

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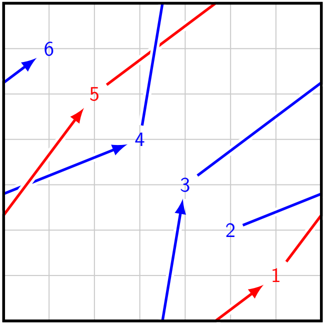

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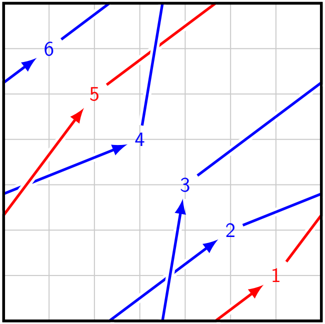

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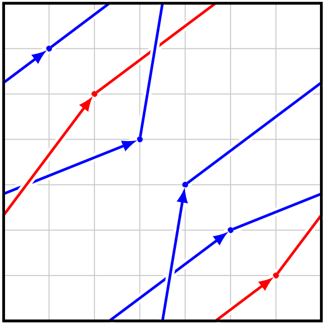

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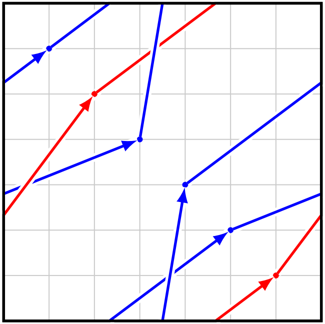

This construction: [G.–Lam '22+]. Related constructions: [G.–Lam '20], [Shende–Treumann–Williams–Zaslow '15], [Fomin–Pylyavskyy–Shustin–Thurston '17], [Casals–Gorsky–Gorsky–Simental '21]

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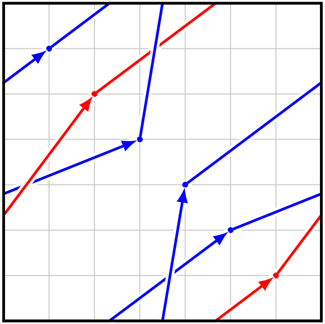
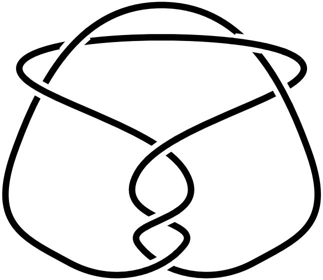


Conclusion

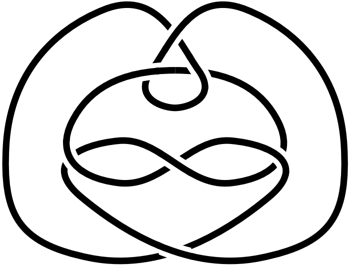
For each permutation $f \in S_n$, get a variety Π_f° and a link $L_f \subseteq \mathbb{R}^3$.

How to tell if two knots/links are isotopic?

One of these knots is not like the others



(A)



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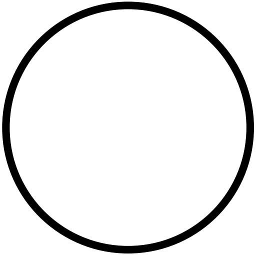

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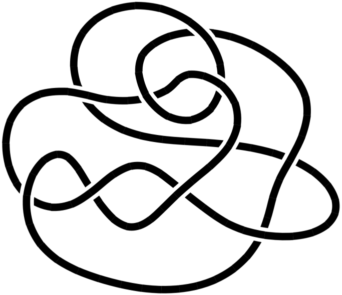

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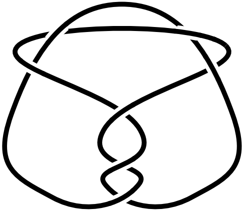
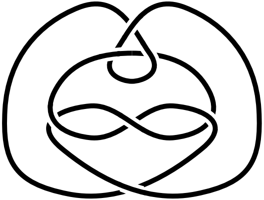


(A) $P(L;a,q) = \frac{q^4-q^3+q^2-q+1}{a^4q^2} + \frac{q^4-q^3+2q^2-q+1}{a^6q^2} - \frac{q^2+1}{a^8q}$



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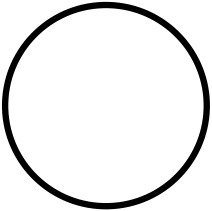

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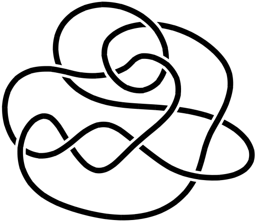

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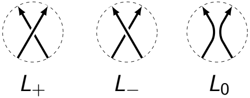

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In this case, Π_f°/T is smooth and $\mathcal{P}(\Pi_f^\circ; q, t) = \left(q^{\frac{1}{2}} + t^{\frac{1}{2}}\right)^{n-1} \cdot \mathcal{P}(\Pi_f^\circ/T; q, t)$.

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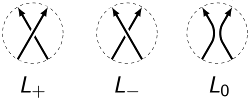

Theorem (G.–Lam)

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Khovanov–Rozansky link homology yields a polynomial $\mathcal{P}_{\mathrm{KR}}(L; a, q, t)$ which specializes to the HOMFLY polynomial $P(L; a, q)$.

Theorem (G.–Lam)

Let $f \in S_n$ be a single cycle. Then

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