Positroids, knots, and q, t-Catalan numbers

Pavel Galashin (Cornell/UCLA)

Yale Mathematics Department Colloquium
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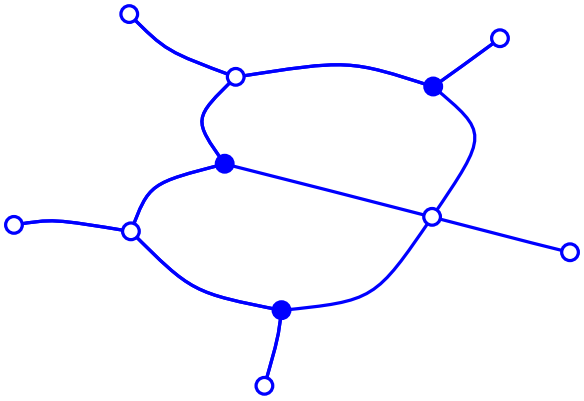
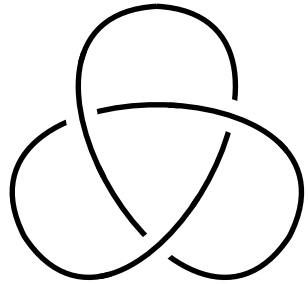
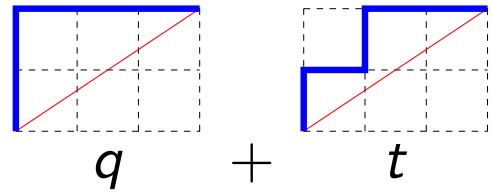
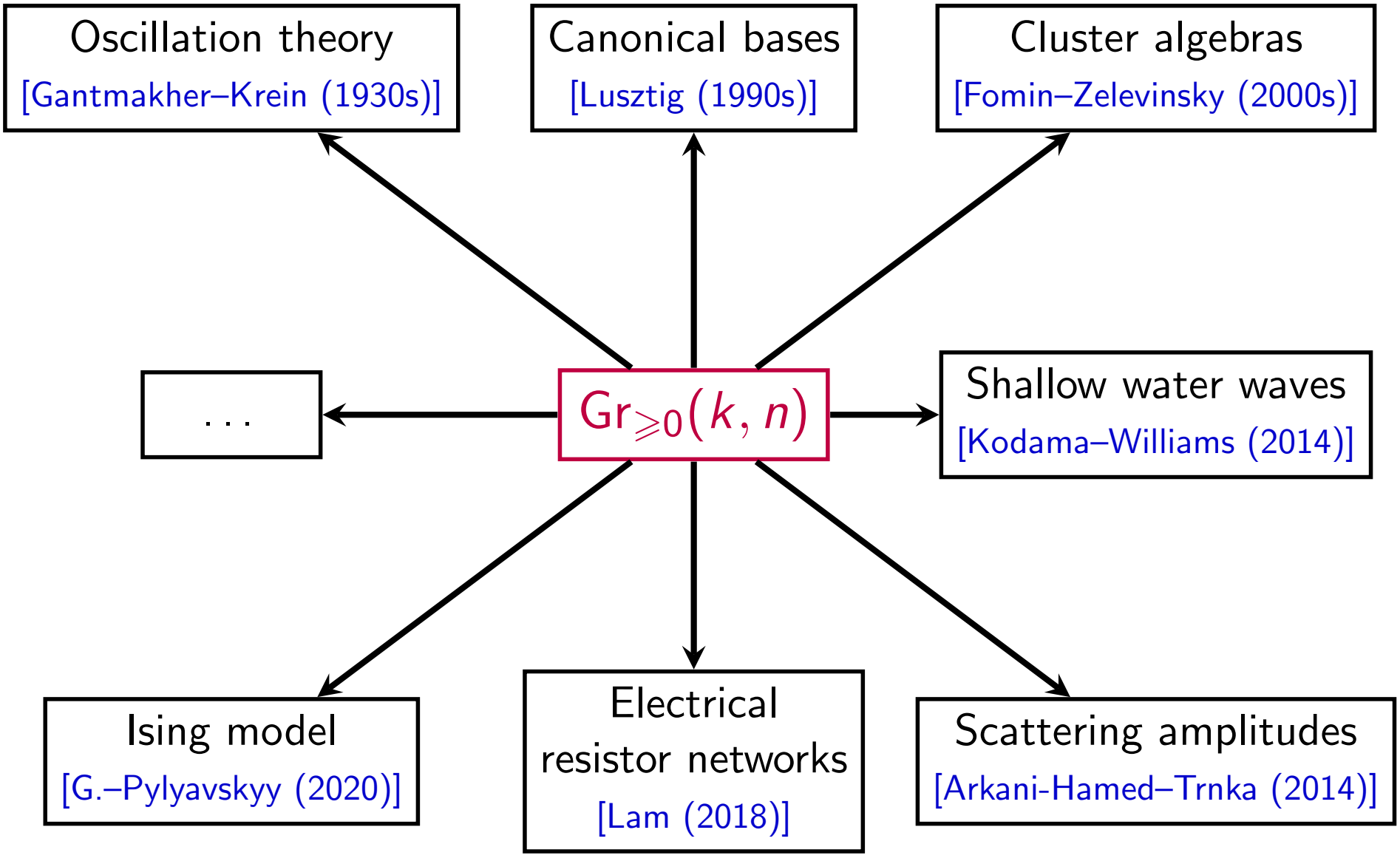
Joint work with Thomas Lam





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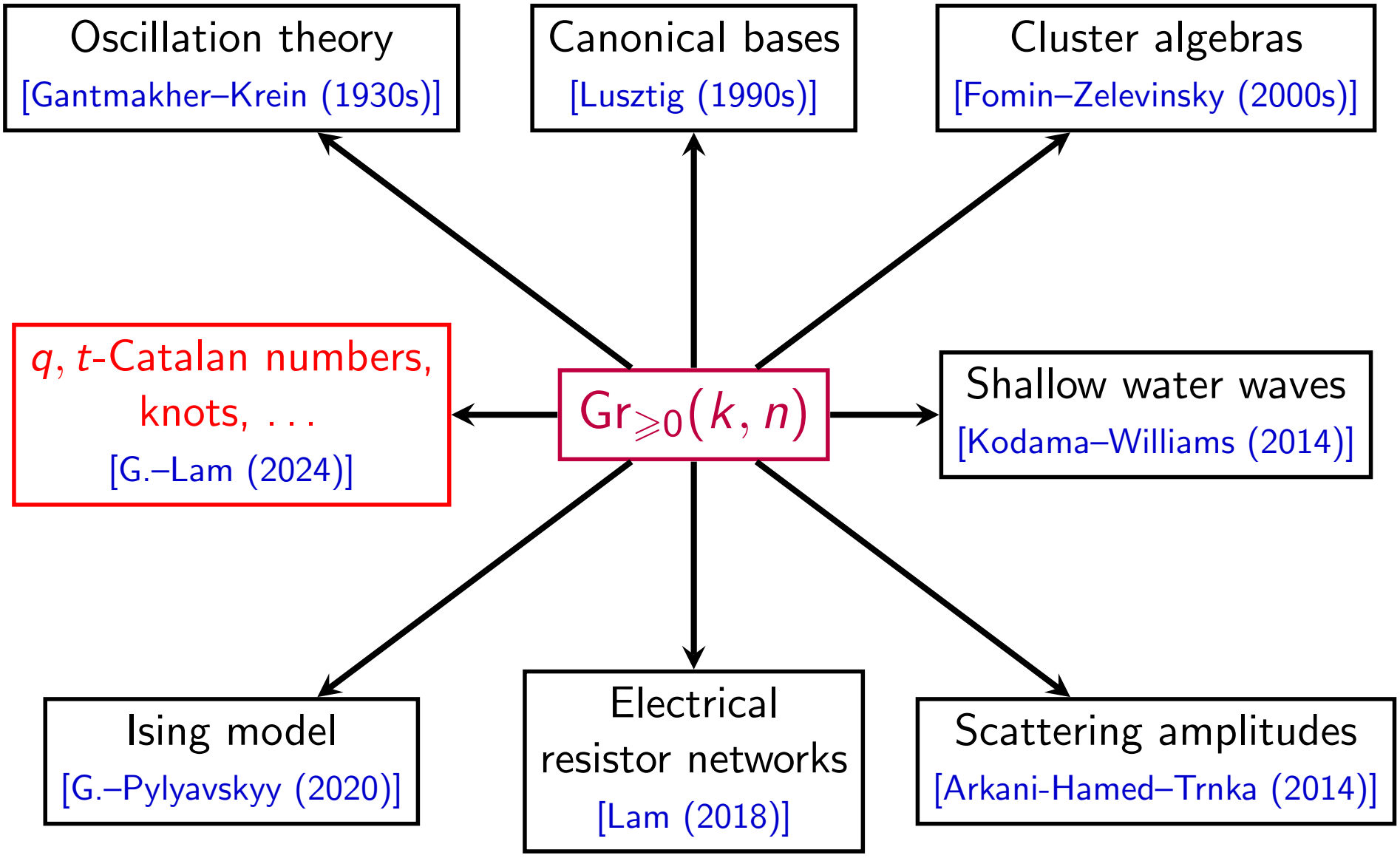
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This is a horrible stratification. [Mnëv (1988)]

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[Pos06] A. Postnikov. Total positivity, Grassmannians, and networks. arXiv:math/0609764.

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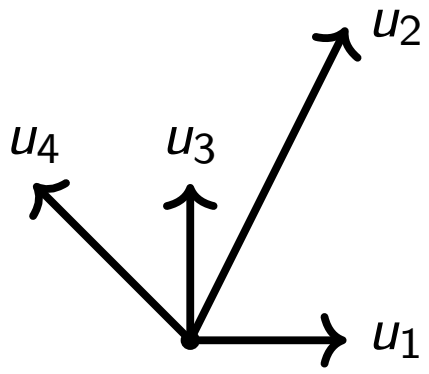

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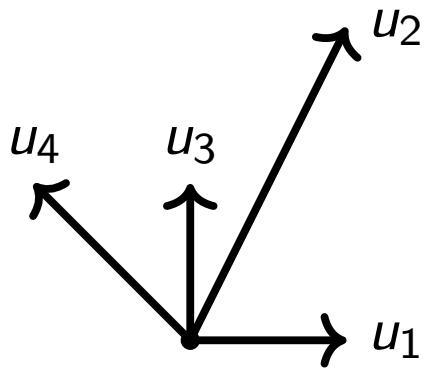

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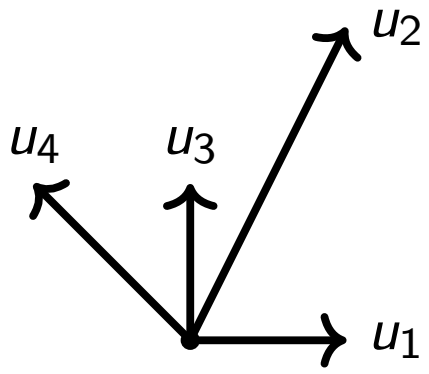

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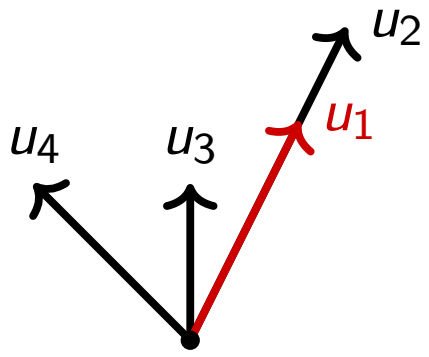

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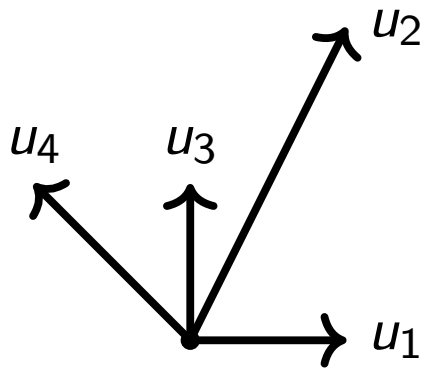

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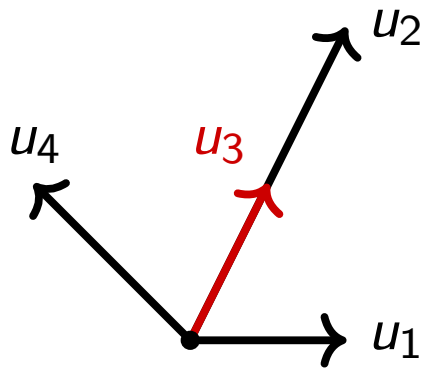

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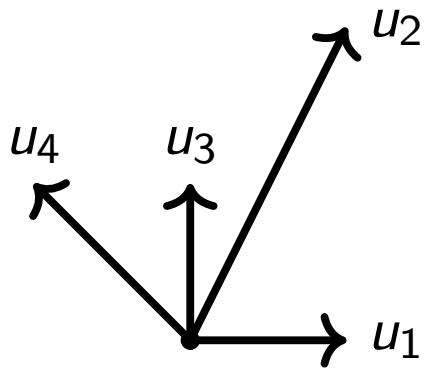

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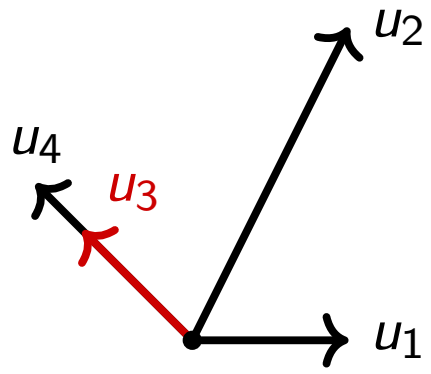

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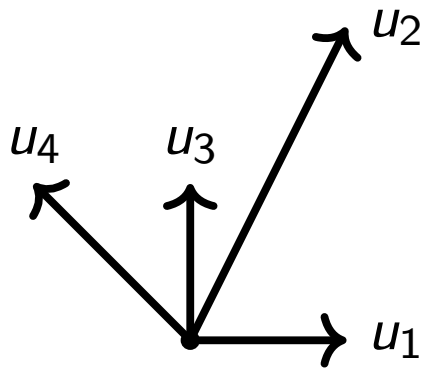

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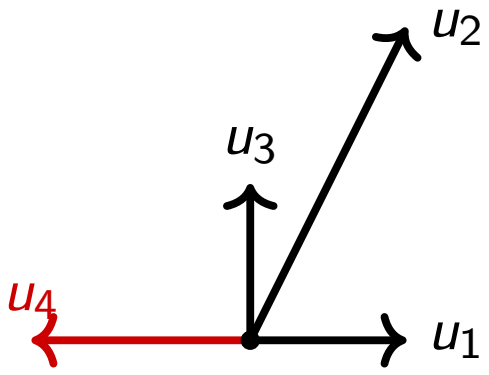

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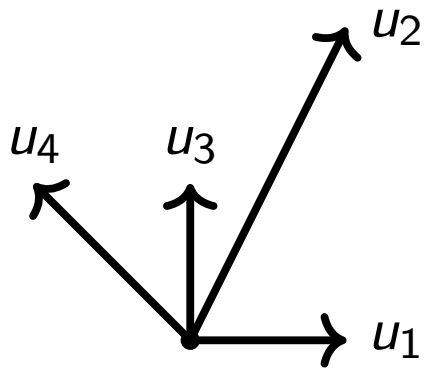

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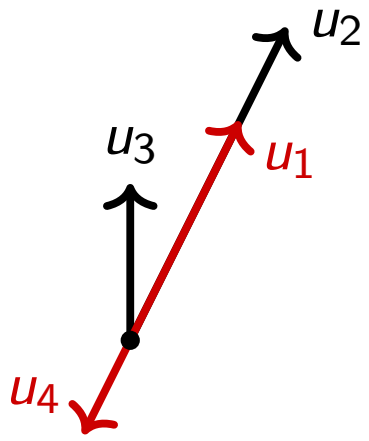

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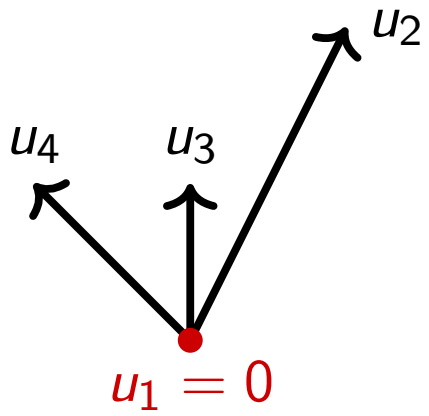

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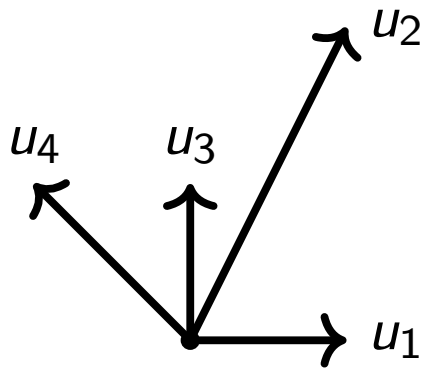

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Question

What is the topology of $\mathrm{Gr}_{\geqslant 0}(k,n)$ and of its boundary cells?

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Theorem (Postnikov (2006))

Each boundary cell (some $\Delta_I > 0$ and the rest $\Delta_J = 0$) is an open ball.

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Theorem (G.–Lam)

- *For $\Pi_{k,n}^\circ$, answer = Catalan numbers and their q-deformations.*

[GL24] P. Galashin and T. Lam. Positroids, knots, and q,t-Catalan numbers. *Duke Math. J.*, 173(11):2117–2195, 2024.

Positroid stratification	$\mathrm{Gr}_{\geqslant 0}(k,n) = \bigsqcup_f \Pi_f^{>0}$	$\subseteq$	$\mathrm{Gr}(k,n;\mathbb{R}) = \bigsqcup_f \Pi_f^\circ$
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Generic stratum	$\Pi_{k,n}^{>0} = \{\text{all } \Delta_I > 0\}$	$\subseteq$	$\Pi_{k,n}^\circ = \left\{ \begin{array}{r} \Delta_{1,\ldots,k} \neq 0, \\ \Delta_{2,\ldots,k+1} \neq 0, \\ \ldots, \\ \Delta_{n,1,\ldots,k-1} \neq 0 \end{array} \right\}$
Topology	Ball		???

Question

Point count $\#\Pi_f^\circ(\mathbb{F}_q) = ?$

Cohomology of $\Pi_f^\circ(\mathbb{C}) = ?$

Euler characteristic of $\Pi_f^\circ(\mathbb{C}) = ?$

Theorem (G.–Lam)

- *For $\Pi_{k,n}^\circ$, answer = Catalan numbers and their q-deformations.*
- *For Π_f°, answer = HOMFLY polynomial/homology of the positroid knot L_f.*

[GL24] P. Galashin and T. Lam. Positroids, knots, and q,t-Catalan numbers. *Duke Math. J.*, 173(11):2117–2195, 2024.

$$\mathrm{Gr}(k, n; \mathbb{F}) := \{W \subseteq \mathbb{F}^n \mid \dim(W) = k\} = \frac{\{\text{full rank } k \times n \text{ matrices over } \mathbb{F}\}}{(\text{row operations})}.$$

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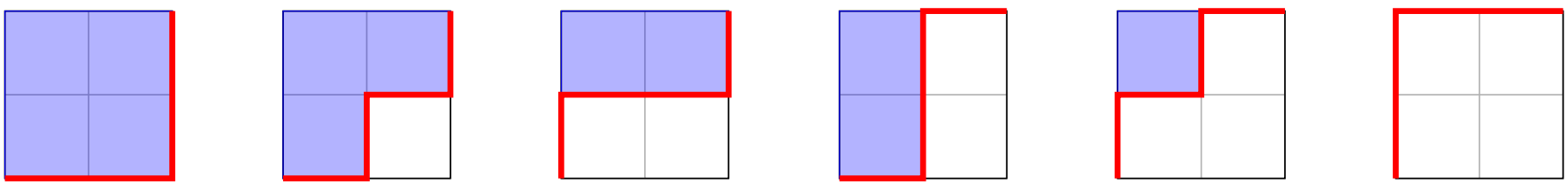

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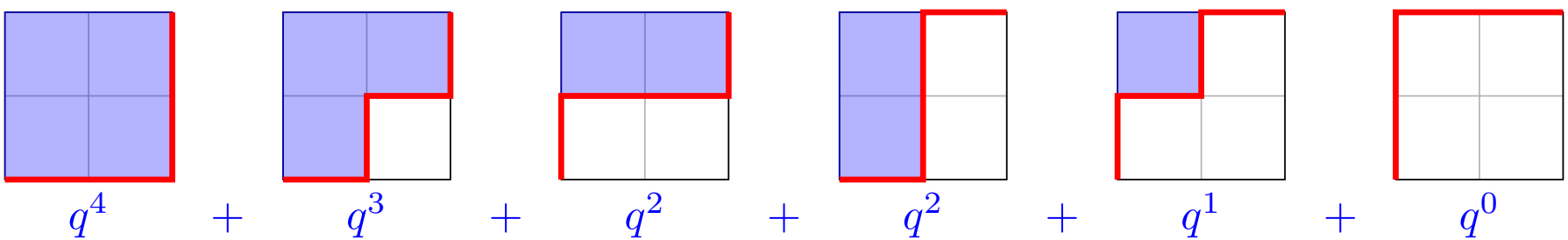

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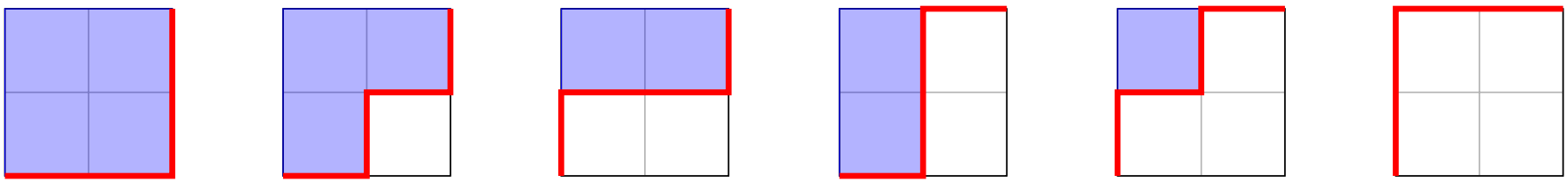

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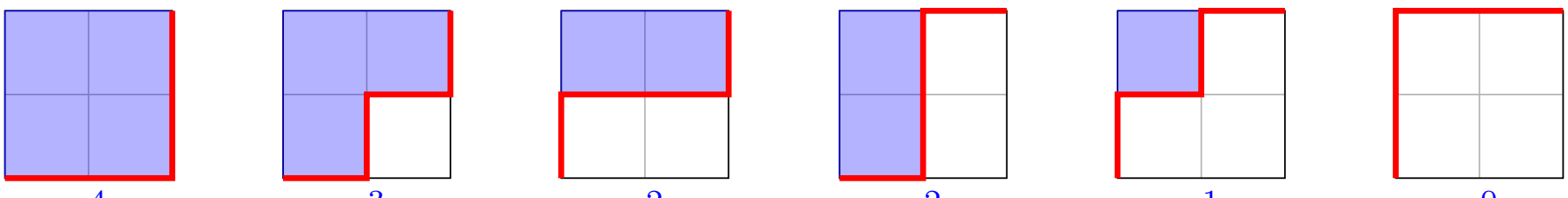

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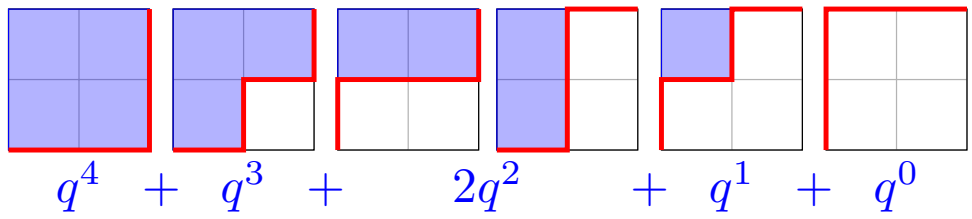

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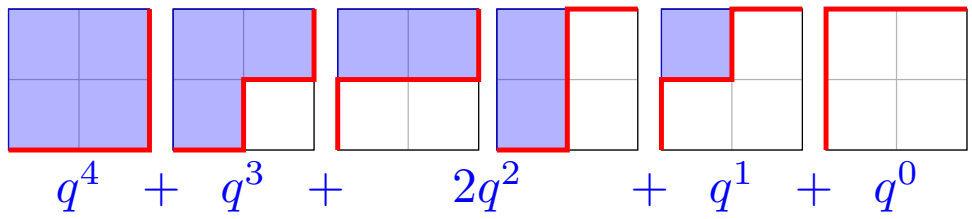

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Coefficients of $\begin{bmatrix} n \\ k \end{bmatrix}_q$ are unimodal (increase up to the middle, then decrease).

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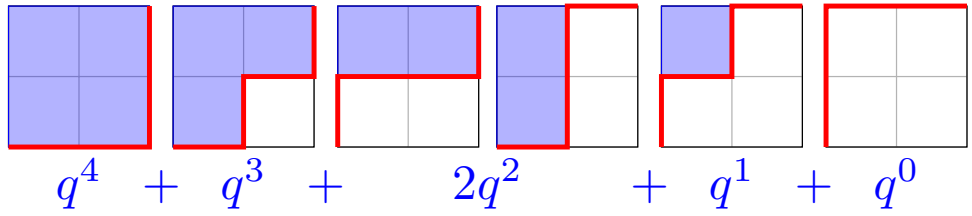

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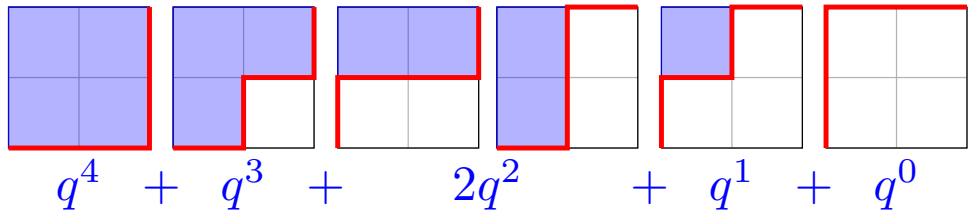

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[**Modern proof:** Follows from the Hard Lefschetz Theorem for $H^\bullet(\mathrm{Gr}(k,n;\mathbb{C}))$.]

- Binomial coefficients:

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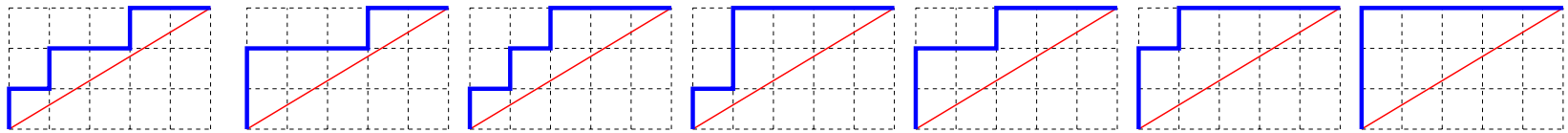
$\mathrm{Dyck}_{3,5} =$ 

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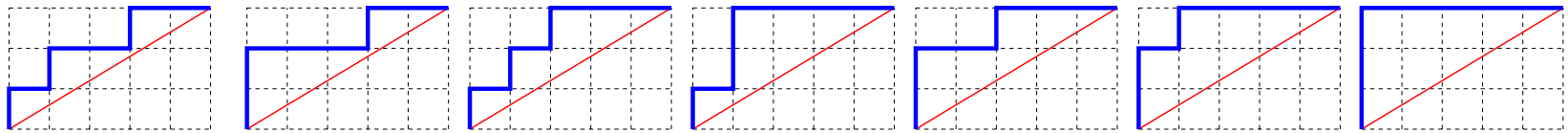
$\mathrm{Dyck}_{3,5} =$ 

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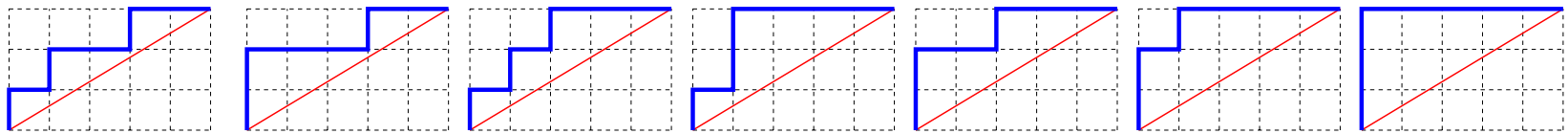
$\mathrm{Dyck}_{3,5} =$ 

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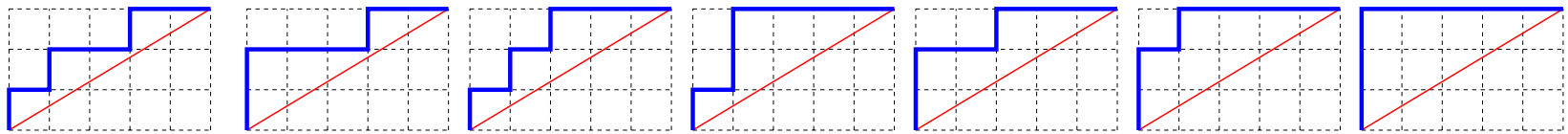
$\mathrm{Dyck}_{3,5} =$ 

$$C'_{3,5}(q) = \frac{1}{[8]_q} \begin{bmatrix} 8 \\ 3 \end{bmatrix}_q = q^8 + q^6 + q^5 + q^4 + q^3 + q^2 + 1.$$

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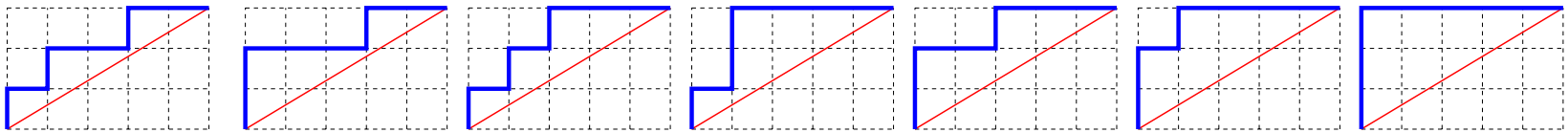
$\mathrm{Dyck}_{3,5} =$ 

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What is "the" q-analog of $C_{k,n-k}$?

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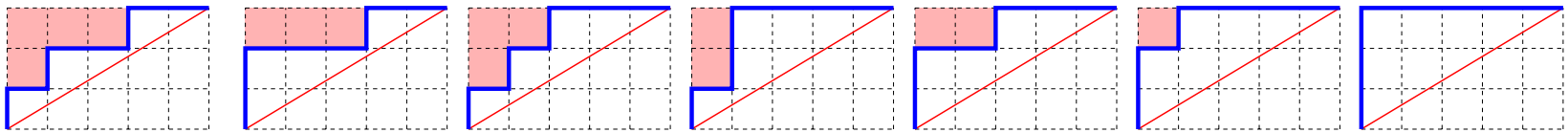
$\mathrm{Dyck}_{3,5} =$ 

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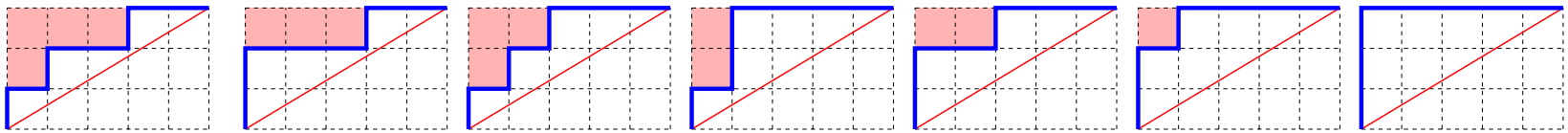
$\mathrm{Dyck}_{3,5} =$ 

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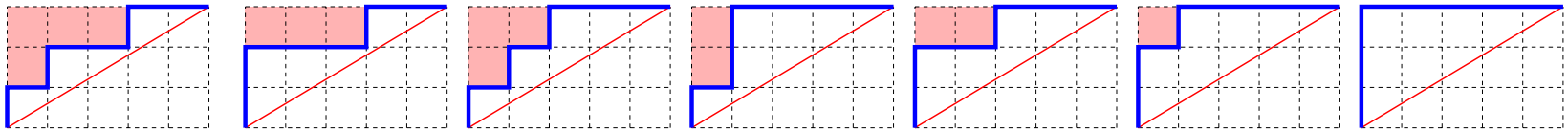
$\mathrm{Dyck}_{3,5} =$ 

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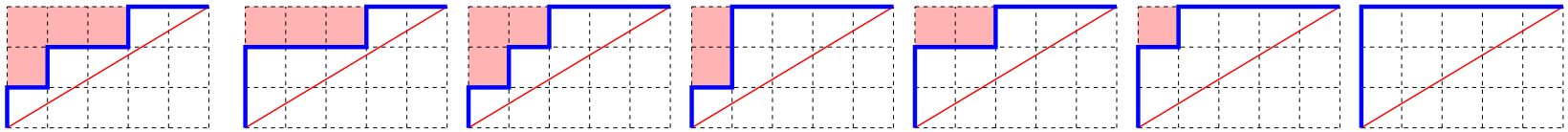
$\mathrm{Dyck}_{3,5} =$ 

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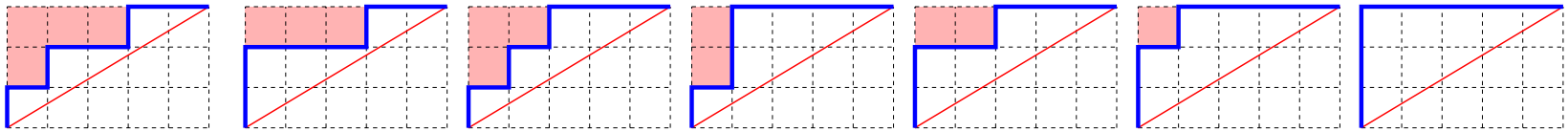
$\mathrm{Dyck}_{3,5} =$ 

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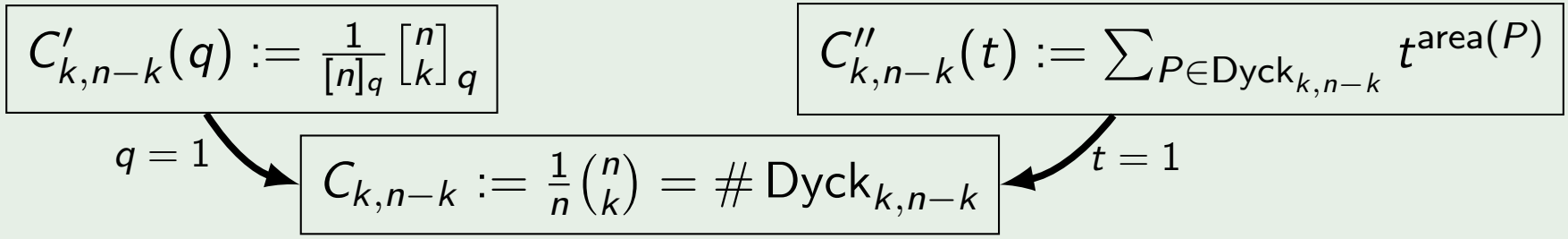

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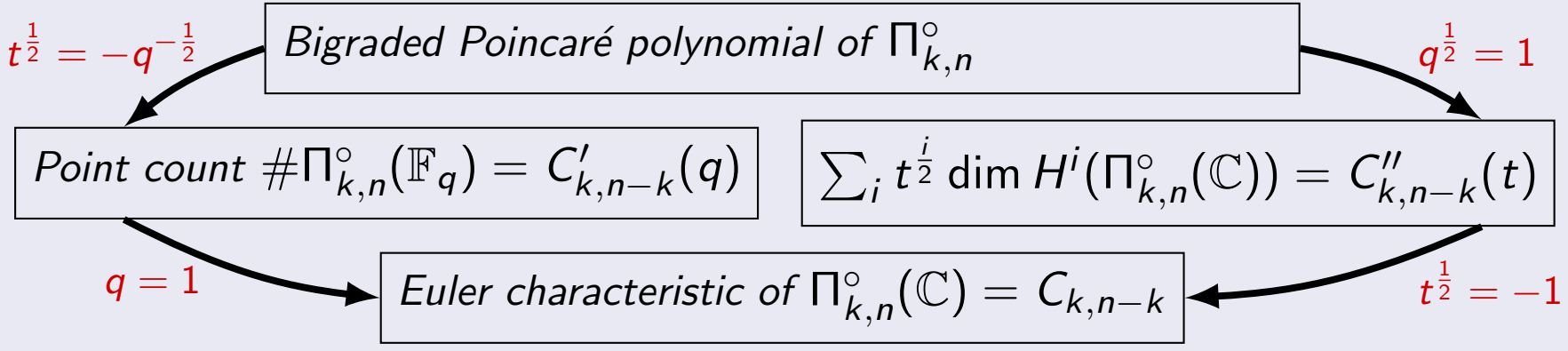

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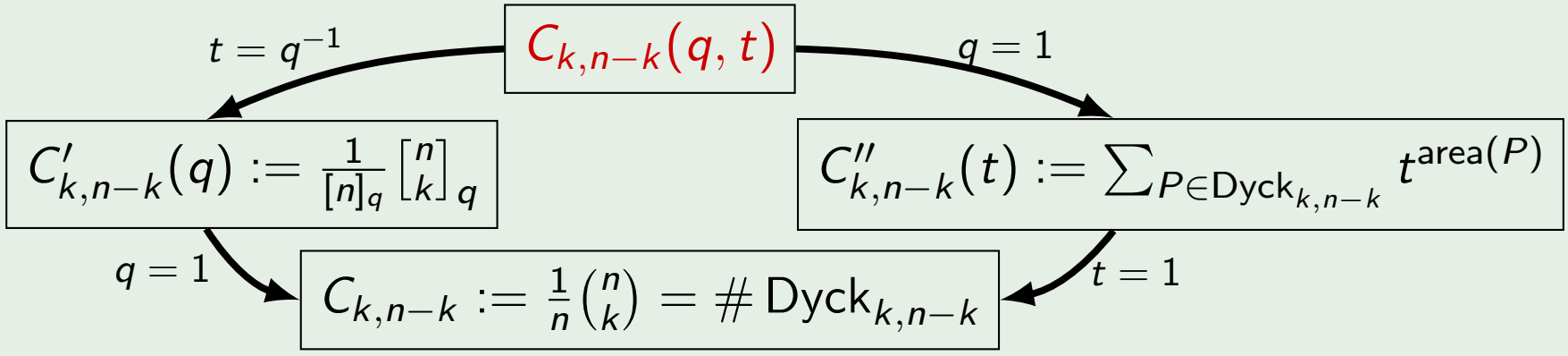

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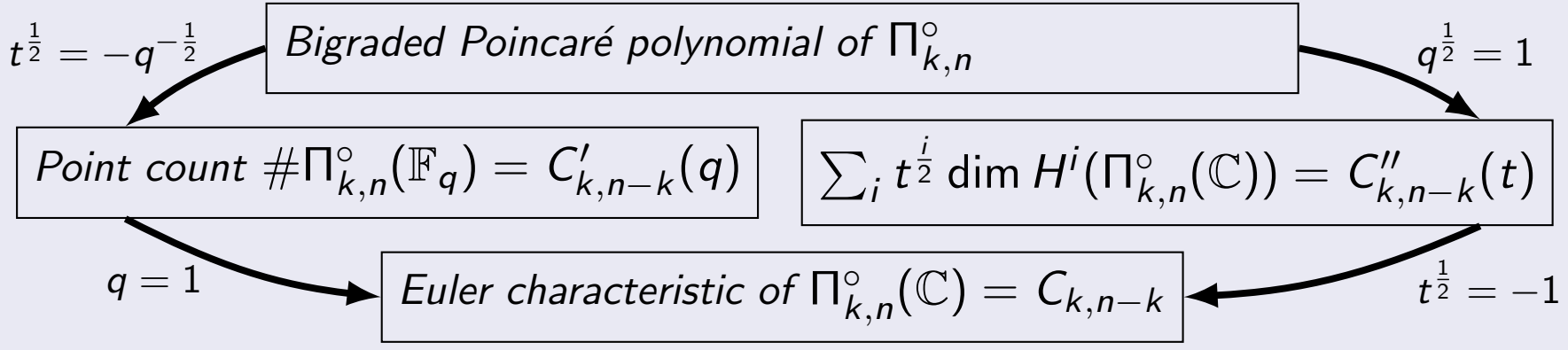

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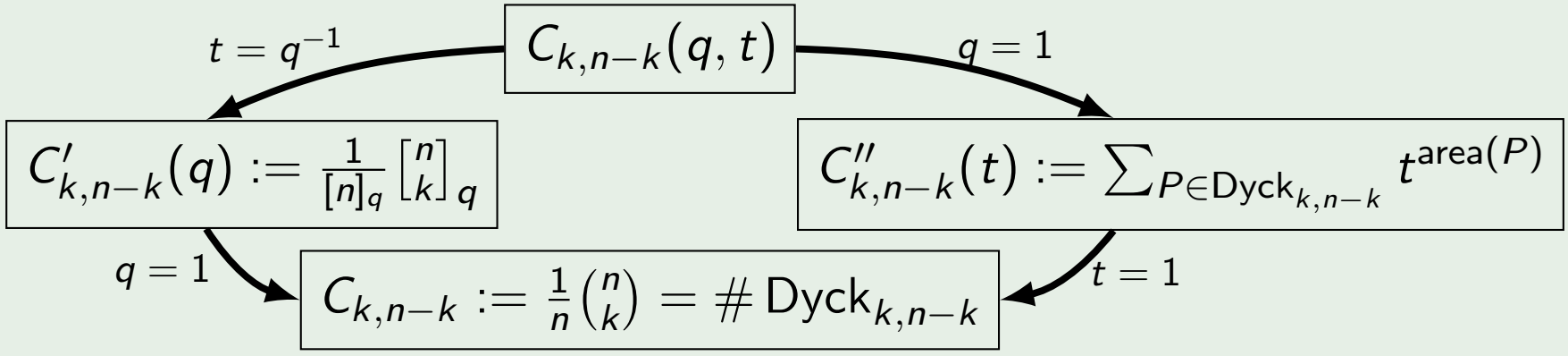

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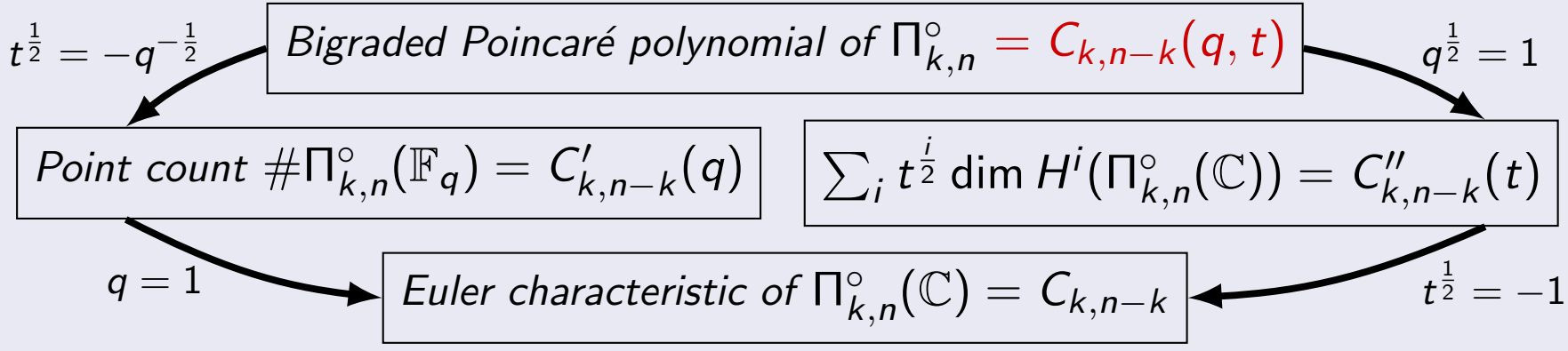

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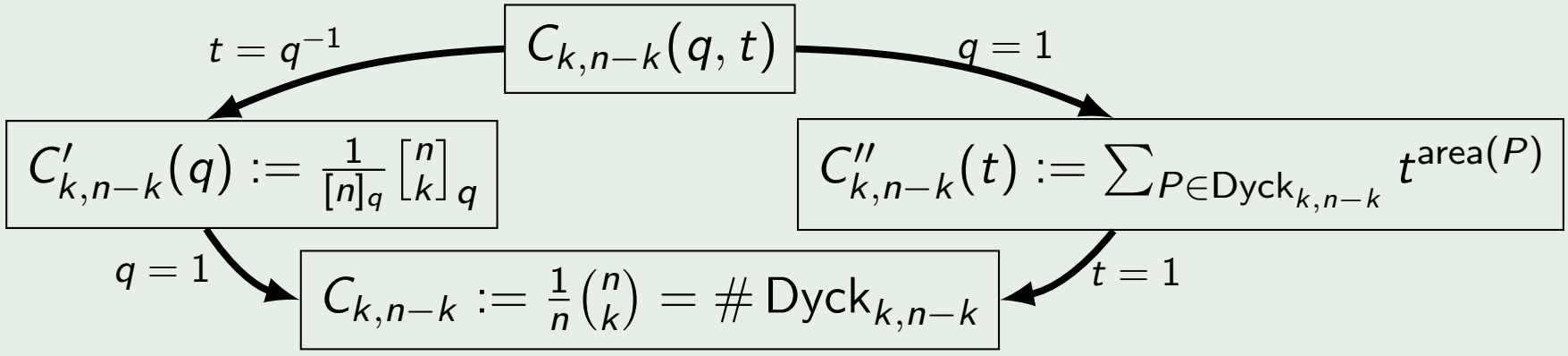

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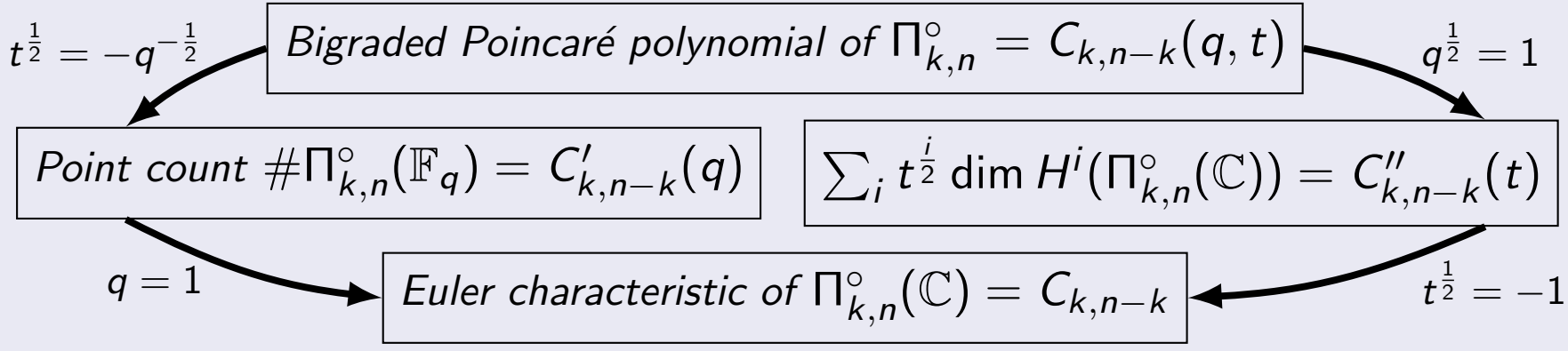

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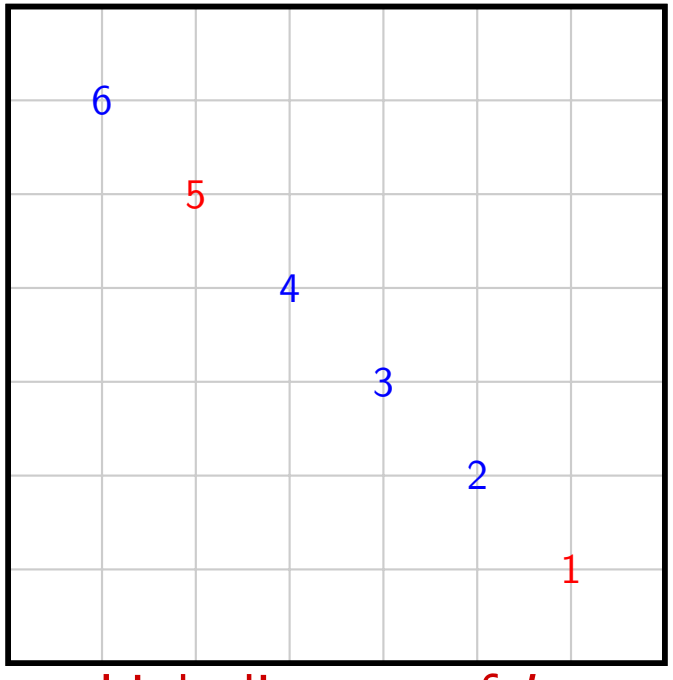

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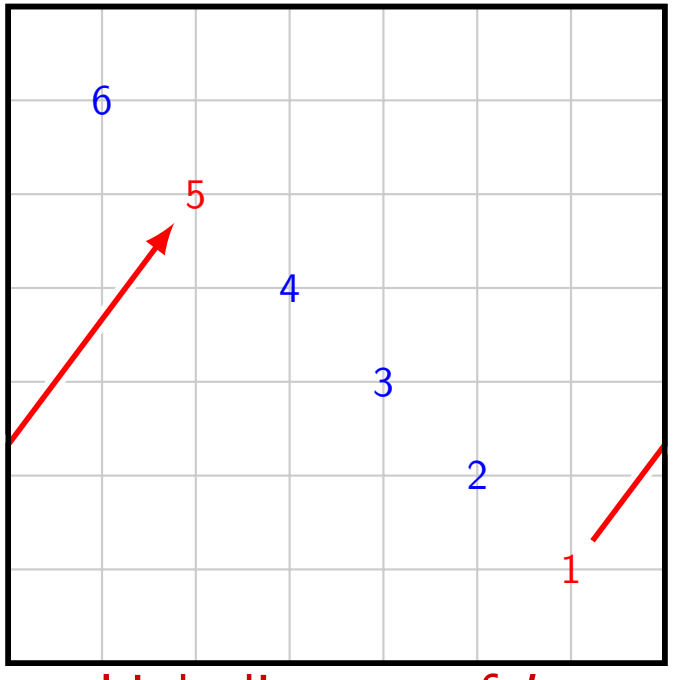

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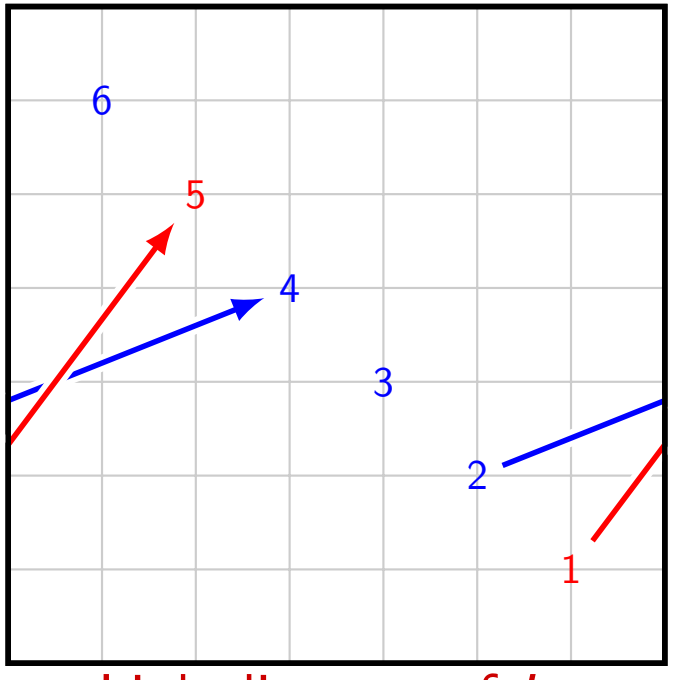

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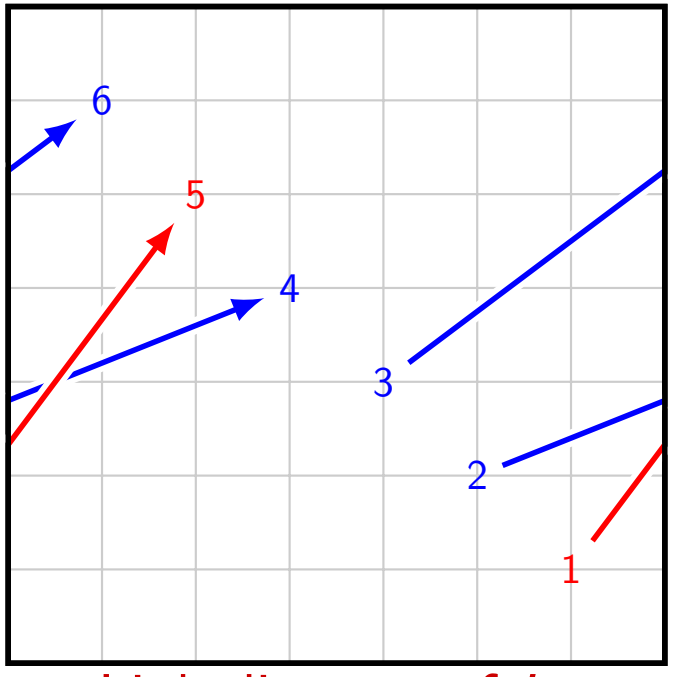

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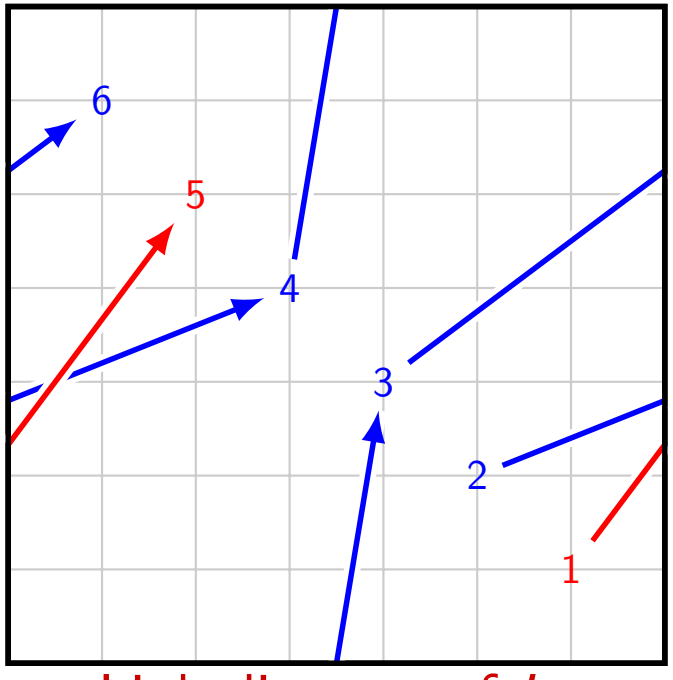

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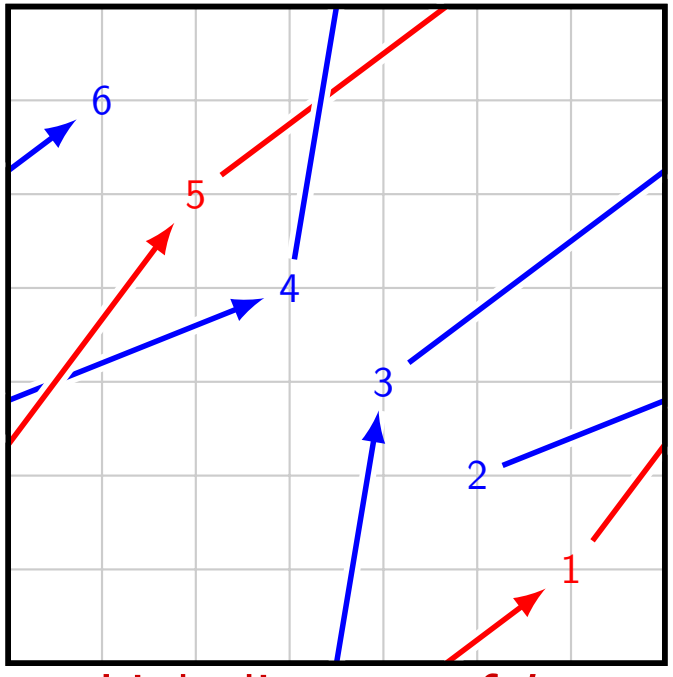

Link diagram of L_f

Positroid links

Positroid stratification: $\mathrm{Gr}(k,n) = \bigsqcup_f \Pi_f^\circ$, where each f is a permutation.

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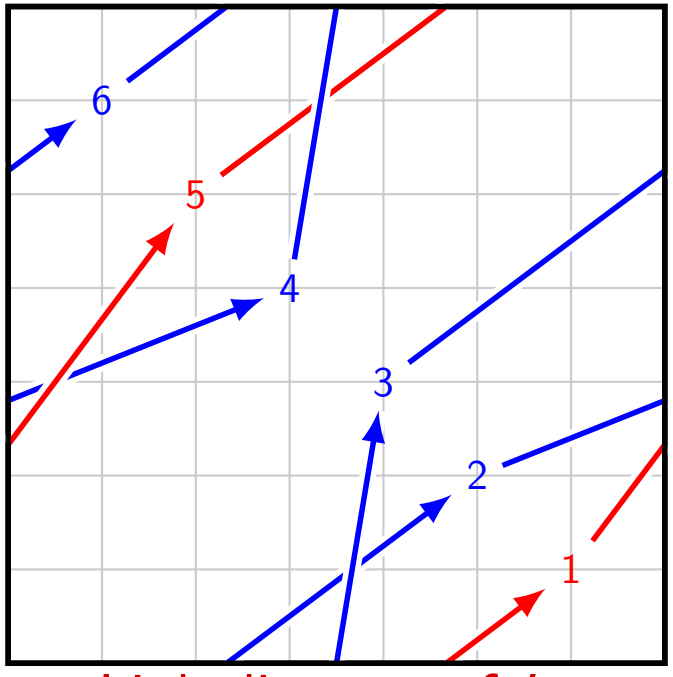

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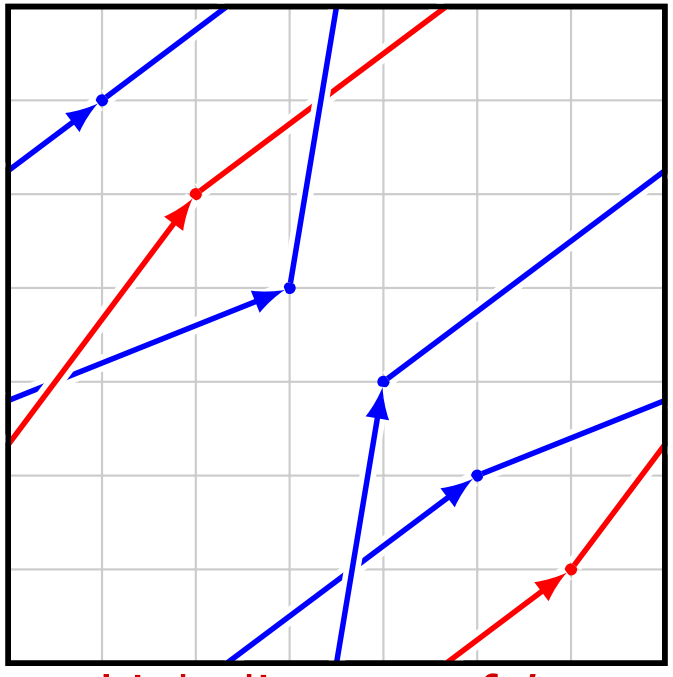

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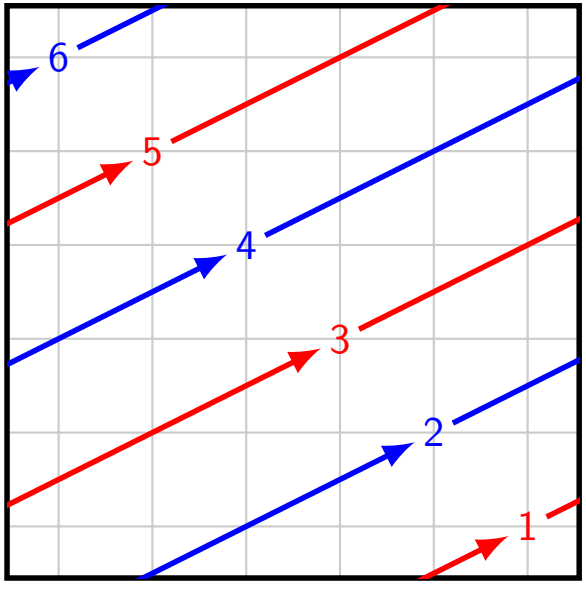

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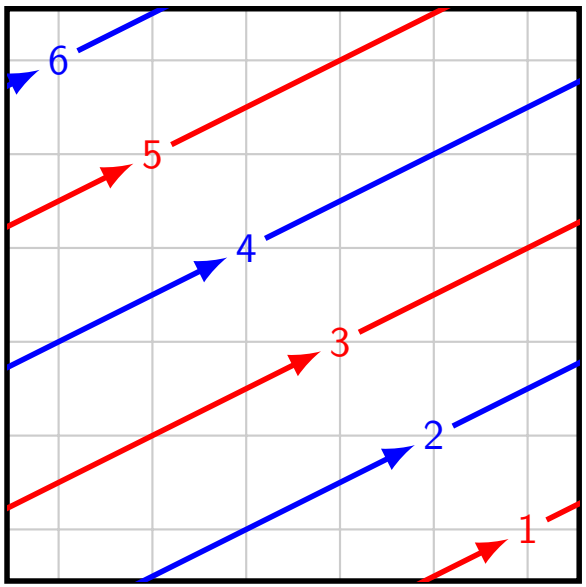

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This construction: [G.–Lam '26+]. Related constructions: [G.–Lam '24], [Shende–Treumann–Williams–Zaslow '15], [Fomin–Pylyavskyy–Shustin–Thurston '17], [Casals–Gorsky–Gorsky–Simental '24]

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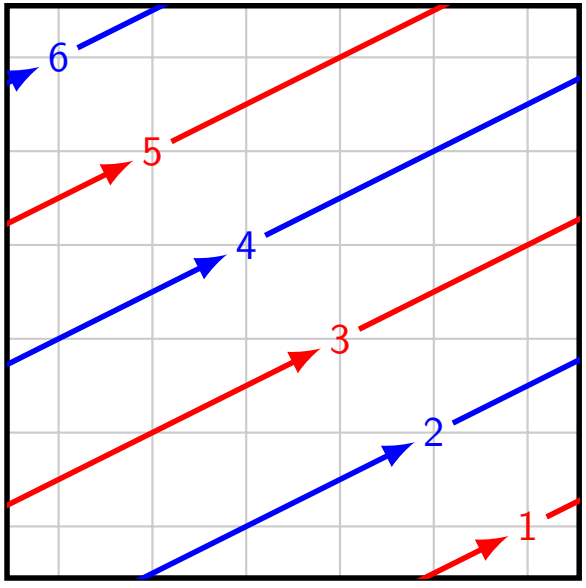

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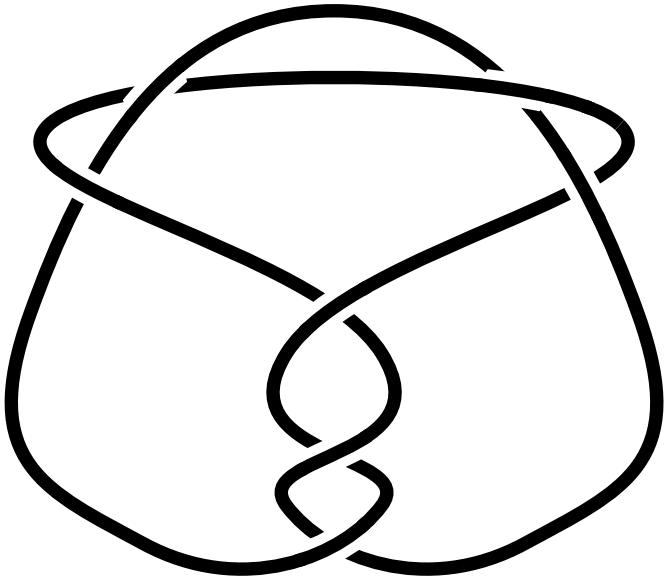
Conclusion

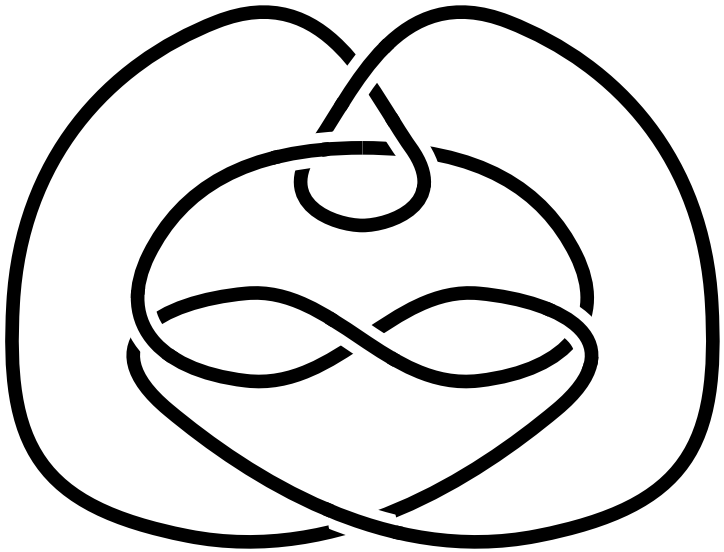
For each permutation $f \in S_n$, get a variety Π_f° and a link $L_f \subseteq \mathbb{R}^3$.

How to tell if two knots/links are isotopic?

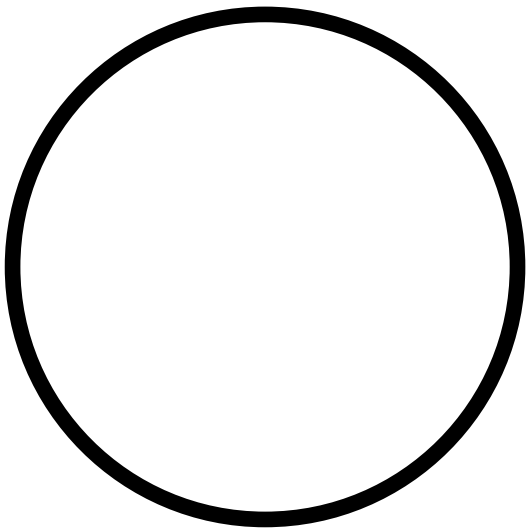
One of these knots is not like the others



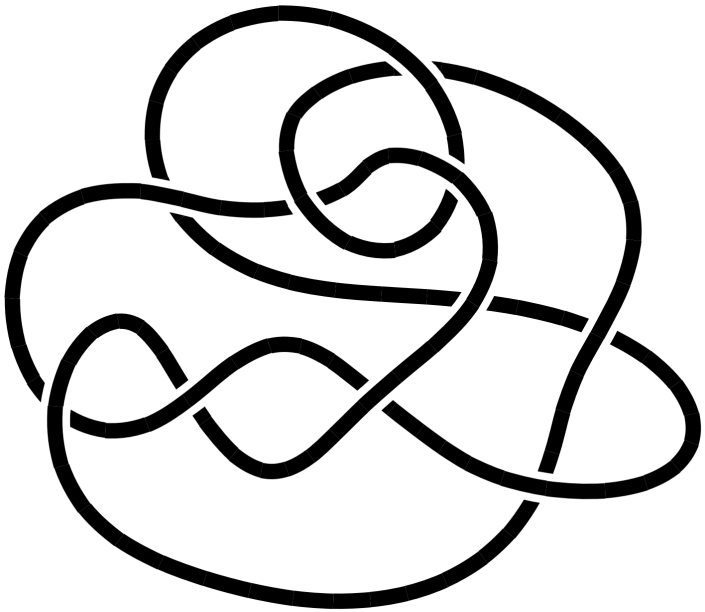
(A)



(B)



(C)



(D)

Given a link L, the HOMFLY polynomial $P(L; a, q)$ is defined by

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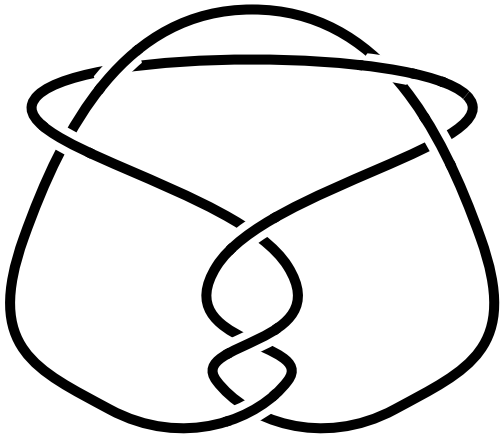
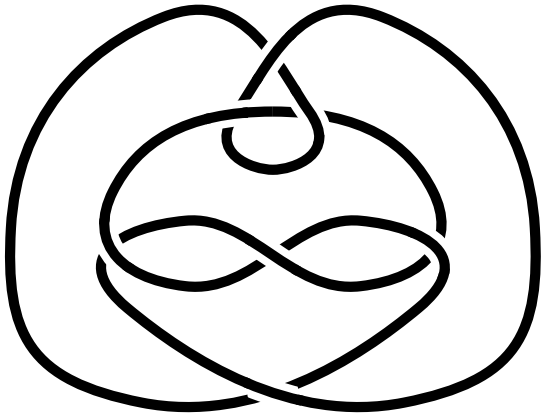
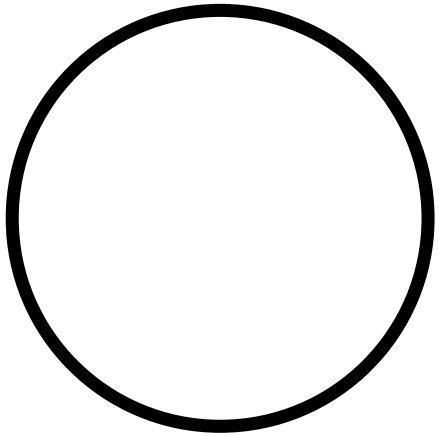


(A) $P(L;a,q) = \frac{q^4-q^3+q^2-q+1}{a^4q^2} + \frac{q^4-q^3+2q^2-q+1}{a^6q^2} - \frac{q^2+1}{a^8q}$



(B) $P(L;a,q) = 1$



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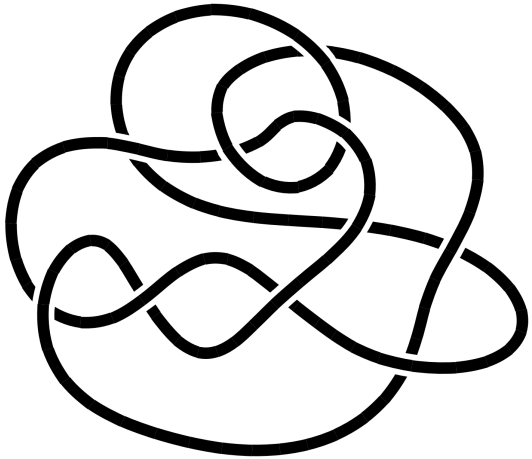

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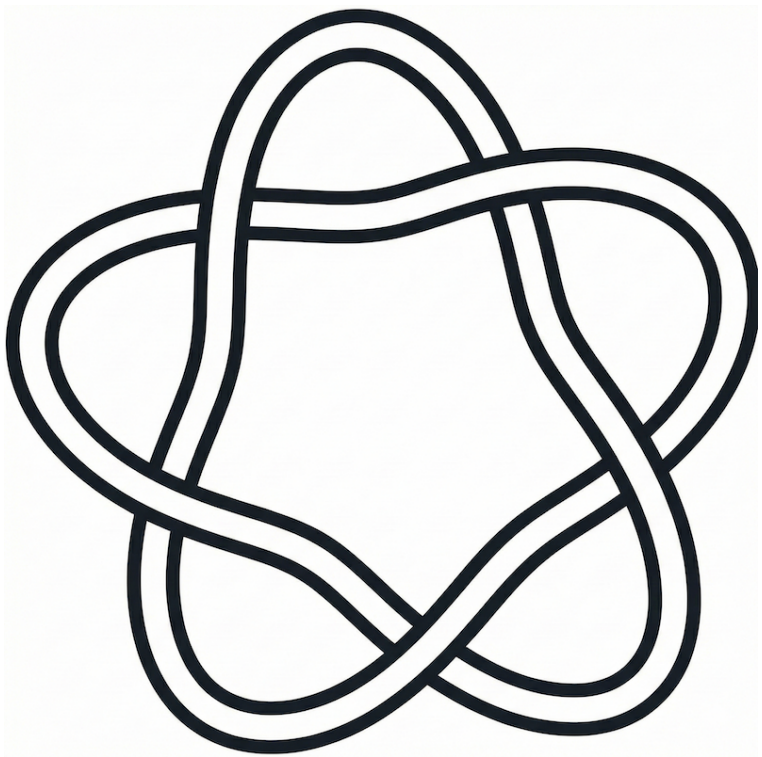

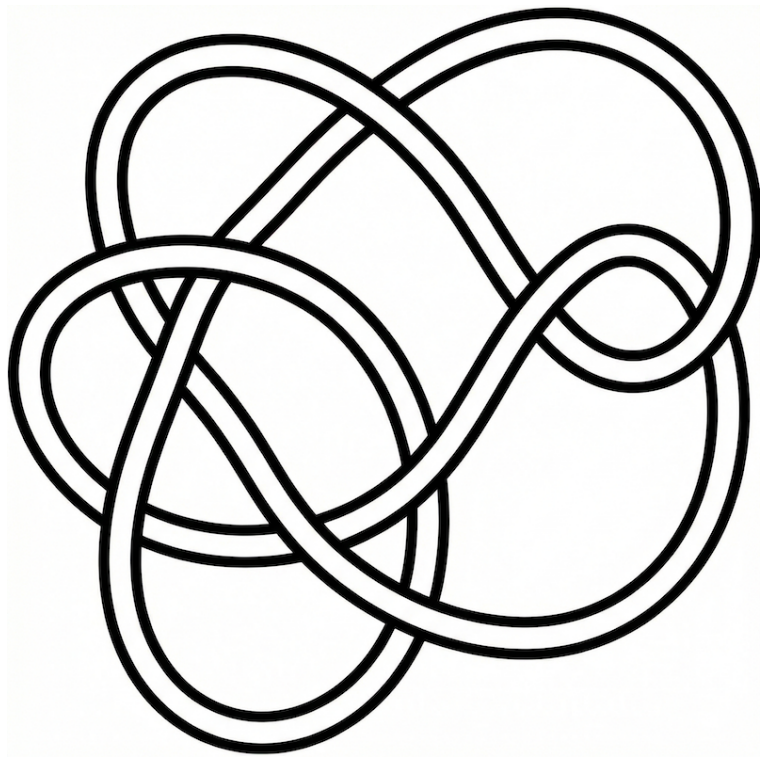

Same HOMFLY/Jones/Alexander polynomials, different $\mathcal{P}_{\mathrm{KR}}(L; a, q, t)$!

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If $f \in S_n$ is a single cycle then **up to a simple multiplicative factor*

Bigraded Poincaré polynomial of $\Pi_f^\circ = \mathrm{Top}_a[\mathcal{P}_{\mathrm{KR}}(L_f; a, q, t)]$

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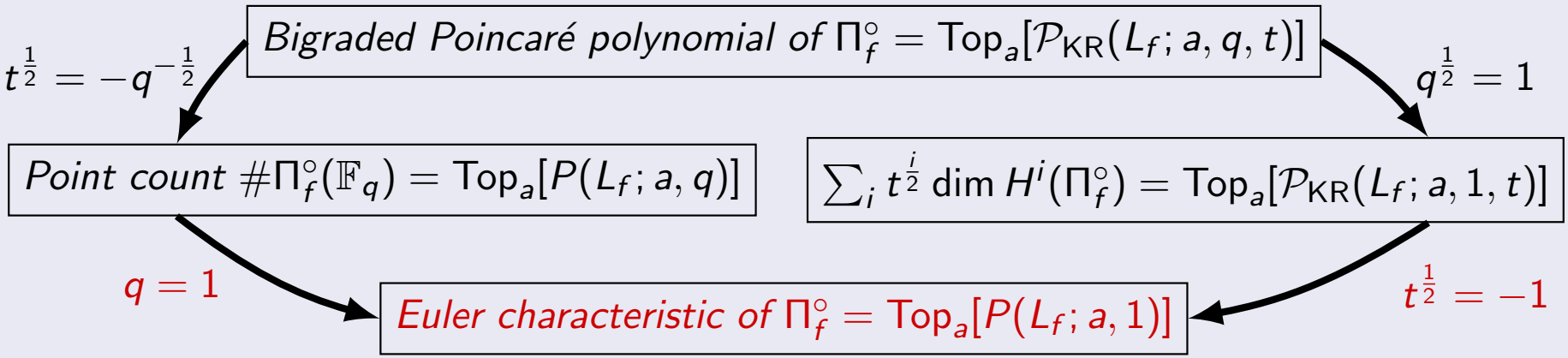

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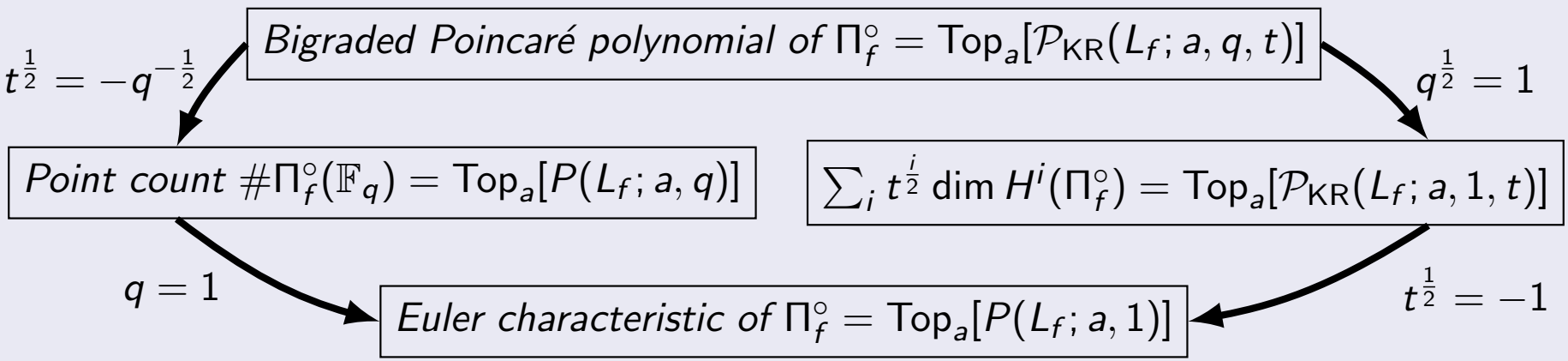

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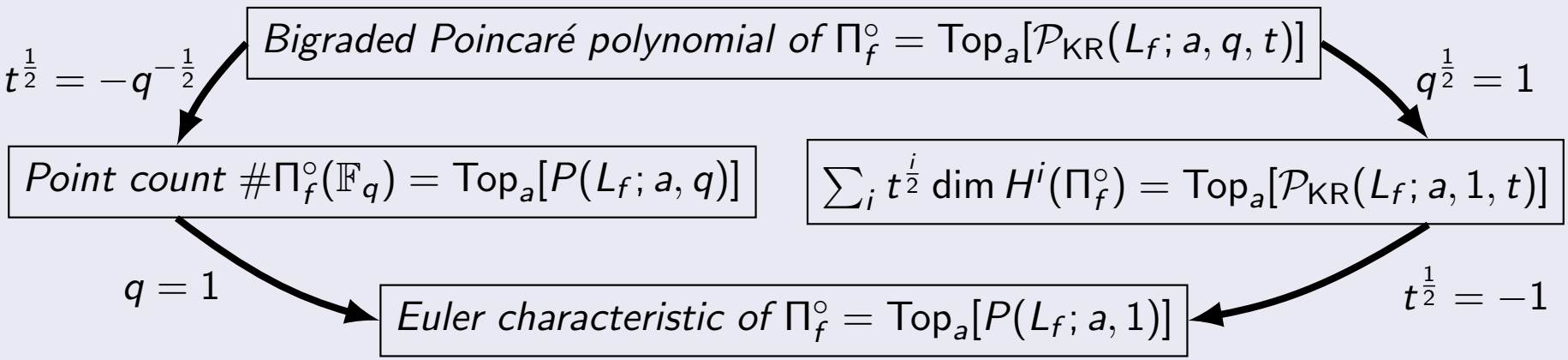

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