

Amplituhedra and origami

Pavel Galashin (Cornell University)

Yale Mathematics Department Colloquium

October 8, 2025

[arXiv:2410.09574](#), [arXiv:25+](#)

≡ Kawasaki's theorem

🌐 6 languages ▾

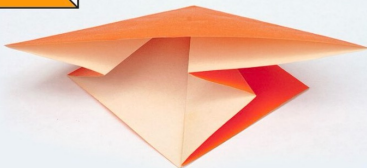
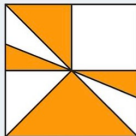
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Kawasaki's theorem or **Kawasaki–Justin theorem** is a [theorem](#) in the [mathematics of paper folding](#) that describes the [crease patterns](#) with a single [vertex](#) that may be folded to form a flat figure. It states that the pattern is flat-foldable if and only if alternately adding and subtracting the angles of consecutive folds around the vertex gives an [alternating sum](#) of zero. Crease patterns with more than one vertex do not obey such a simple criterion, and are [NP-hard](#) to fold.



In this example, the alternating sum of angles (clockwise from the bottom) is $90^\circ - 45^\circ + 22.5^\circ - 22.5^\circ + 45^\circ - 90^\circ + 22.5^\circ - 22.5^\circ = 0^\circ$. Since it adds to zero, the crease pattern may be flat-folded. 

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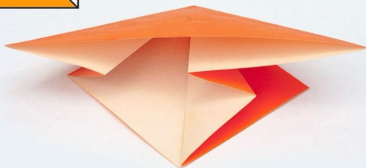
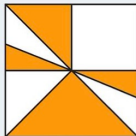
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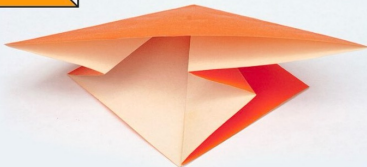
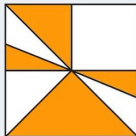
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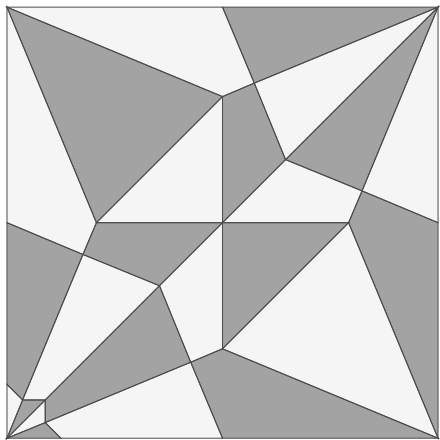


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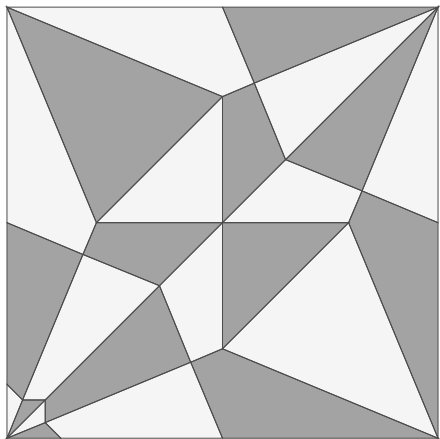


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Origami crease patterns (OCP)

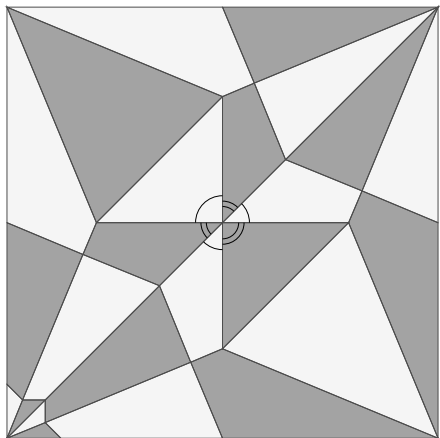


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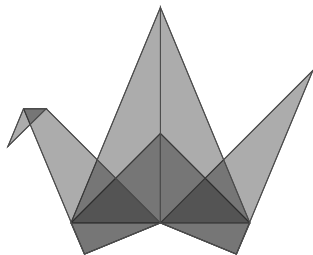
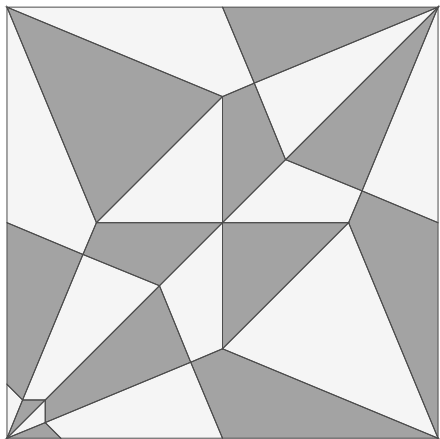
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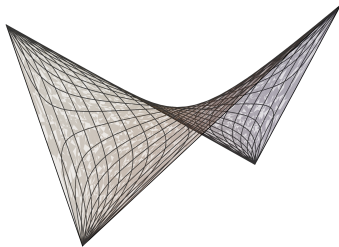
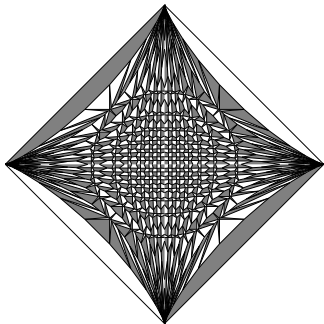
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- **Origami map** $\mathcal{O}$: isometry on each face preserving/reversing the orientations of white/black faces.

Convergence results for dimer model observables on OCPs:
[Kenyon–Lam–Ramassamy–Russkikh '18], [Chelkak–Laslier–Russkikh '21]



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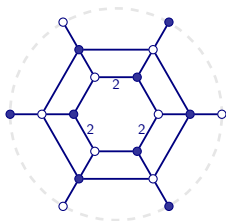
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Conjecture (OCP Existence Conjecture [KLRR '18], [CLR '21])

Every weighted planar bipartite graph Γ can be realized geometrically* as a planar dual of an embedded OCP.



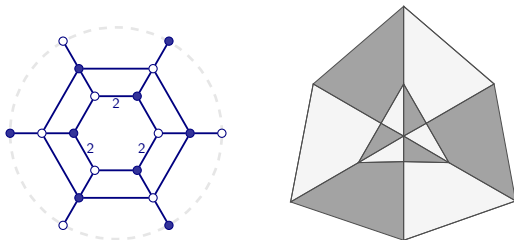
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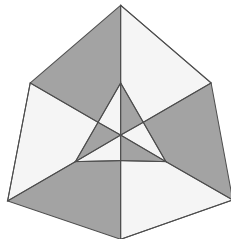
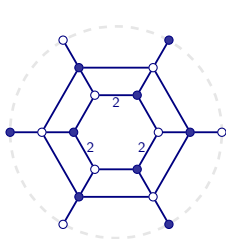
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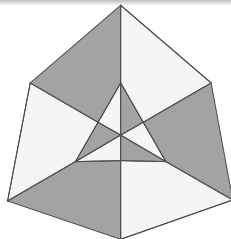
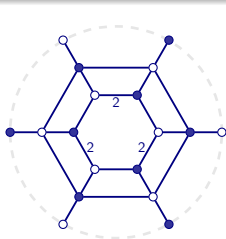
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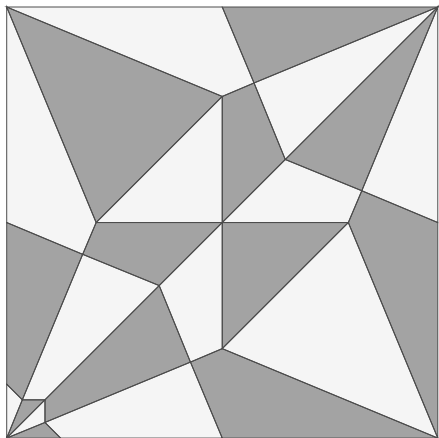
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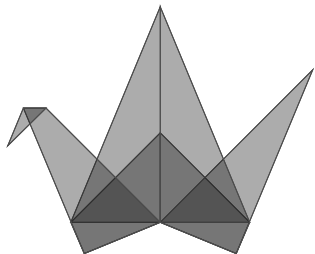
Proof.

Total positivity + physics of scattering amplitudes. □



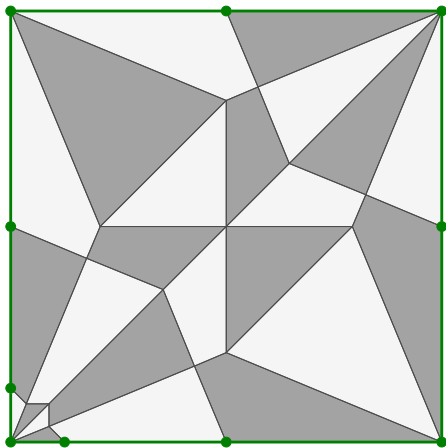


Kami plane

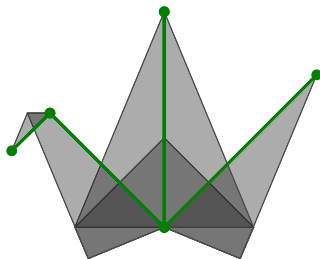


Origami plane

- OCP + its origami folding = 2-dimensional discrete PL surface in $\mathbb{R}^{2,2} \cong \mathbb{C} \times \mathbb{C}$.

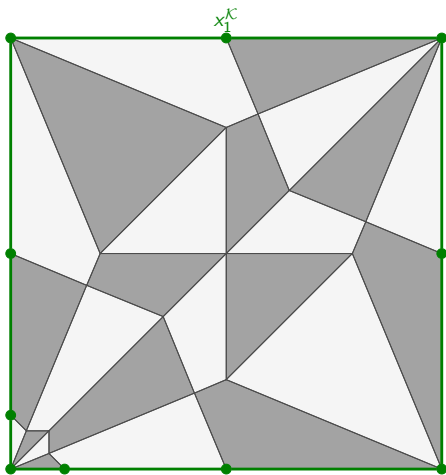


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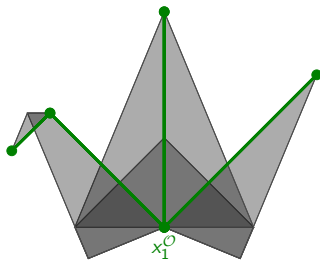


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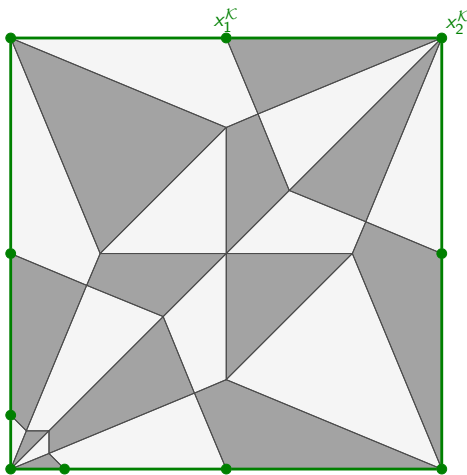


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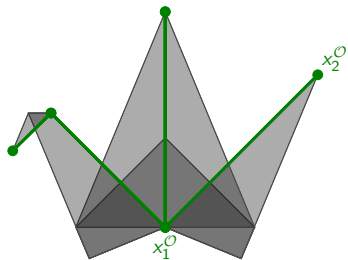


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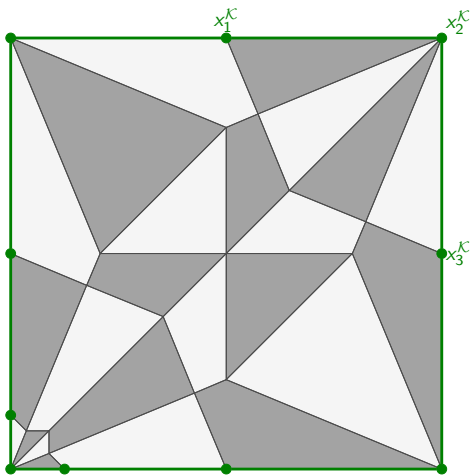


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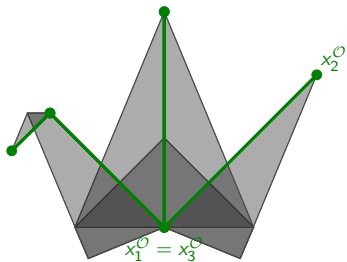


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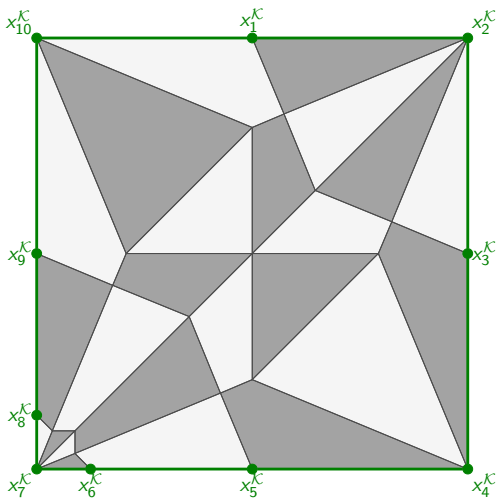


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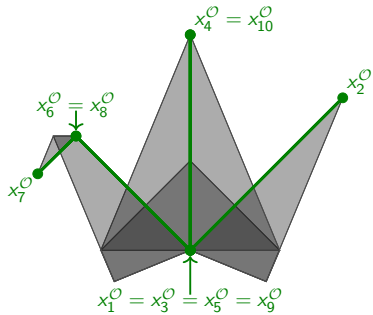


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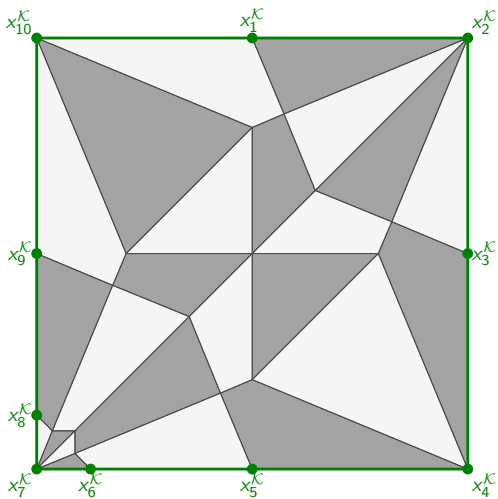


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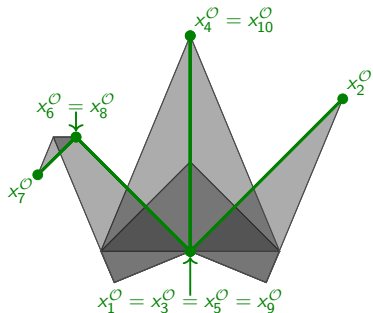


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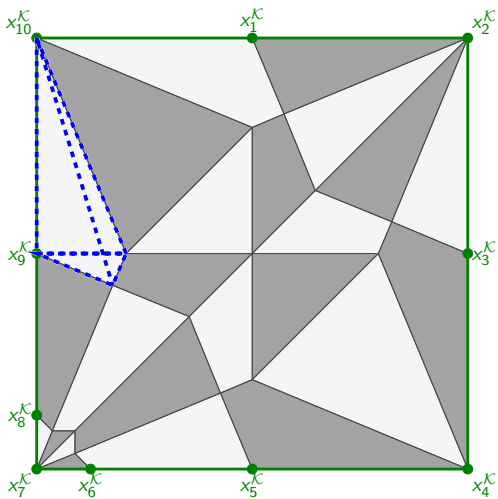


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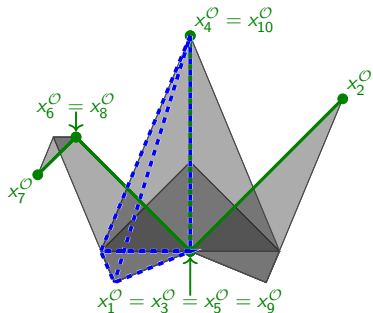


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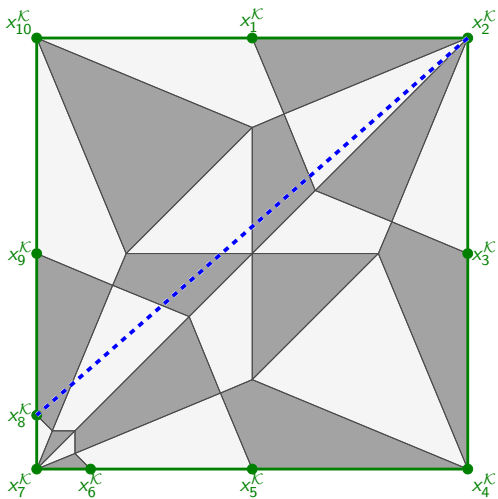


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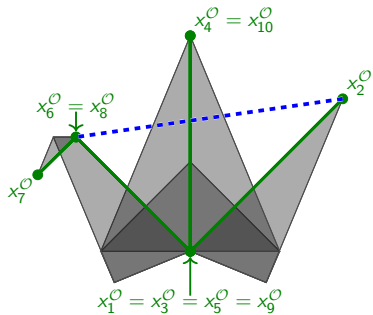


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- Origami map does not increase distances: $(x_i - x_j)^2 \geq 0$, i.e., $|x_i^K - x_j^K| \geq |x_i^O - x_j^O|$.

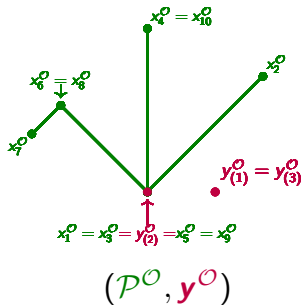
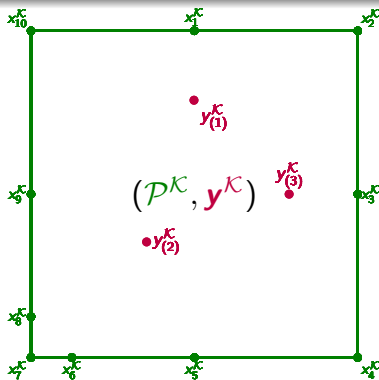
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Definition

Let $L \geq 0$. An L -punctured polygon is a pair $(\mathcal{P}, \mathbf{y})$, where

$\mathcal{P} = (x_1, x_2, \dots, x_n) \in (\mathbb{R}^{2,2})^n$ and $\mathbf{y} = (y_{(1)}, y_{(2)}, \dots, y_{(L)}) \in (\mathbb{R}^{2,2})^L$, such that



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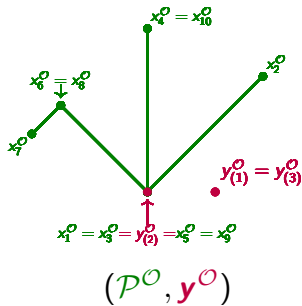
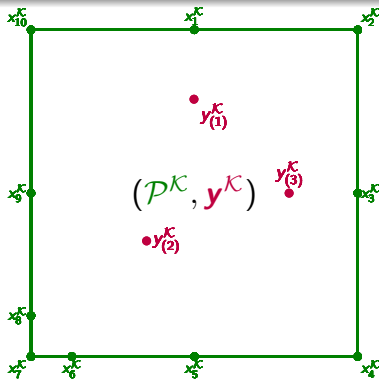
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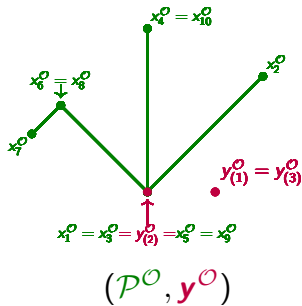
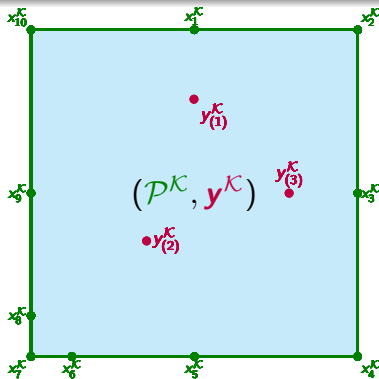
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each $y_{(\rho)}^K$ is located inside the polygon $\mathcal{P}^K = (x_1^K, x_2^K, \dots, x_n^K)$.



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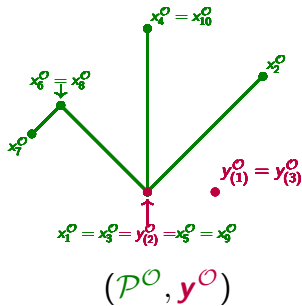
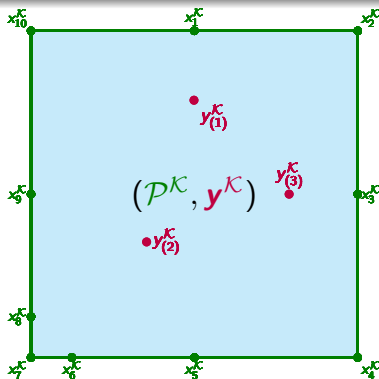
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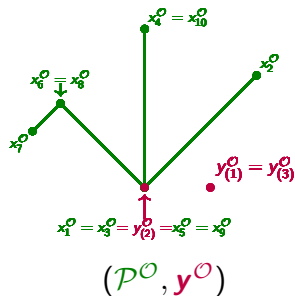
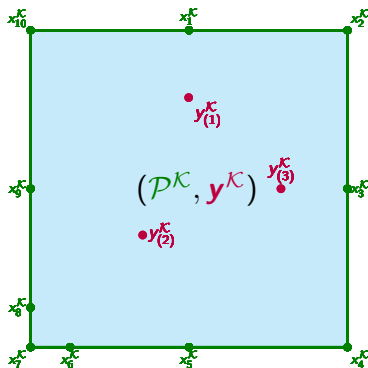
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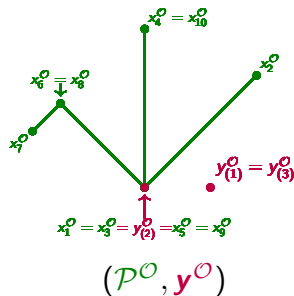
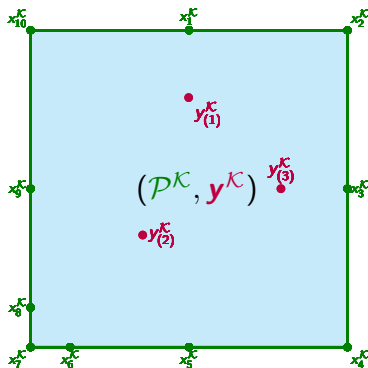
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Problem: Find an OCP with boundary $\mathcal{P}^K$ such that the origami map sends $x_i^K \mapsto x_i^O$ and $y_{(\rho)}^K \mapsto y_{(\rho)}^O$ for all i, ρ .



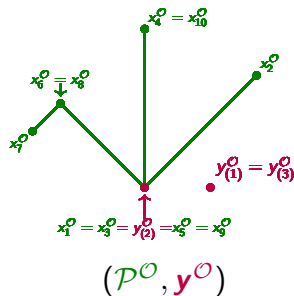
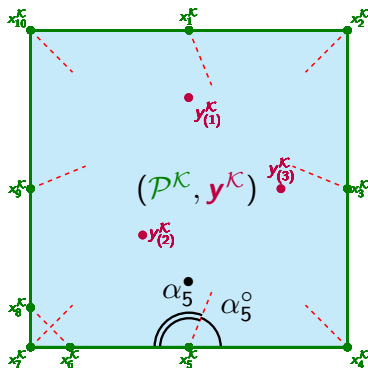
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- Recover white/black angle sums $(\alpha_i^\circ, \alpha_i^\bullet)$ from the geometry of $\mathcal{P}$:

$$\alpha_i^\circ + \alpha_i^\bullet = \alpha_i^K \quad \text{and} \quad \alpha_i^\circ - \alpha_i^\bullet \equiv \alpha_i^O \pmod{2\pi}.$$

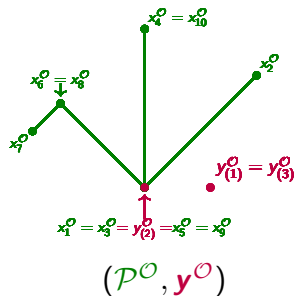
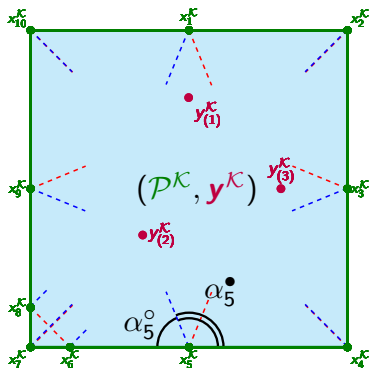


Problem: Find an OCP with boundary $\mathcal{P}^K$ such that the origami map sends $x_i^K \mapsto x_i^O$ and $y_{(\rho)}^K \mapsto y_{(\rho)}^O$ for all i, ρ .

- Recover white/black angle sums (α_i^O, α_i^K) from the geometry of $\mathcal{P}$:

$$\alpha_i^O + \alpha_i^K = \alpha_i^K \quad \text{and} \quad \alpha_i^O - \alpha_i^K \equiv \alpha_i^O \pmod{2\pi}.$$

- Red folding ray splits the angle α_i^K into angles (α_i^K, α_i^O) .

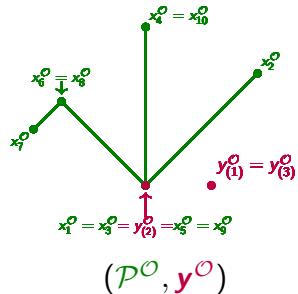
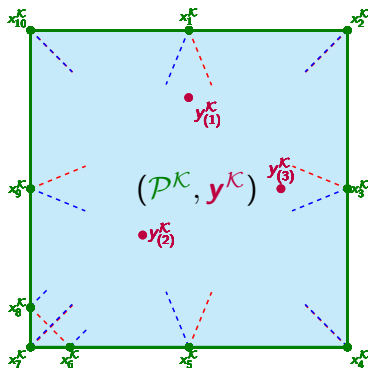


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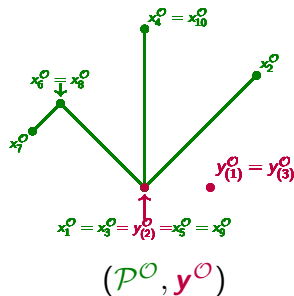
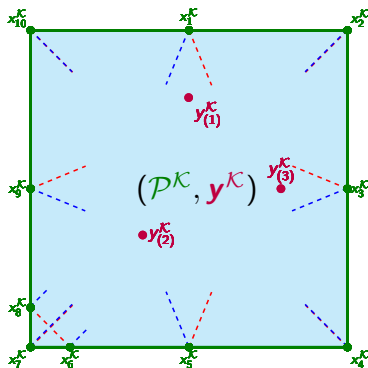
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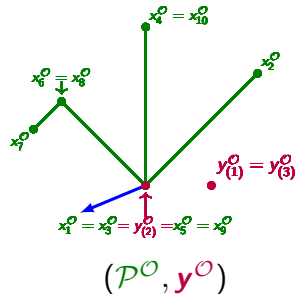
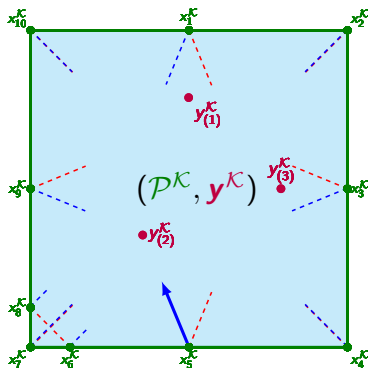
Origami Reconstruction Algorithm



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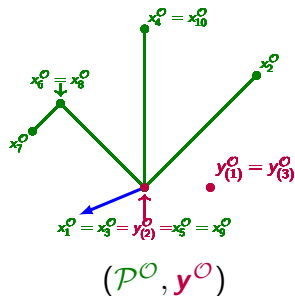
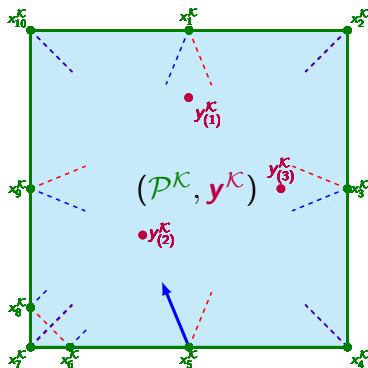
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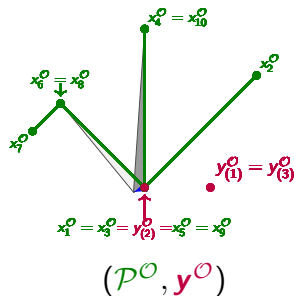
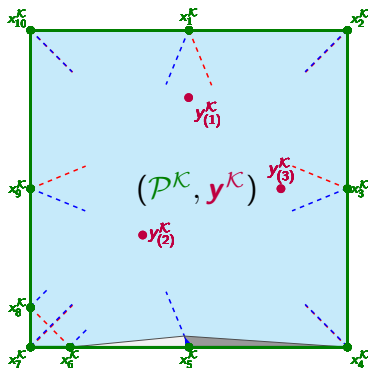
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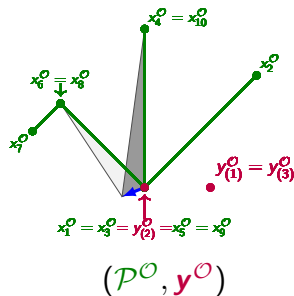
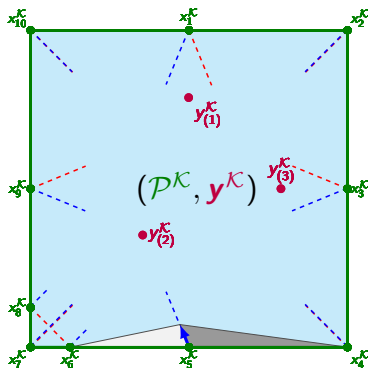
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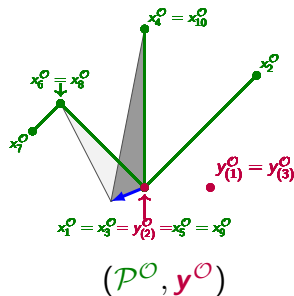
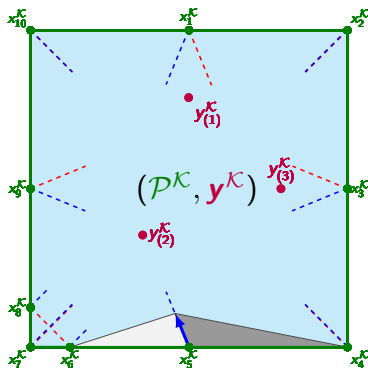
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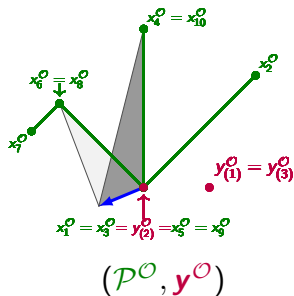
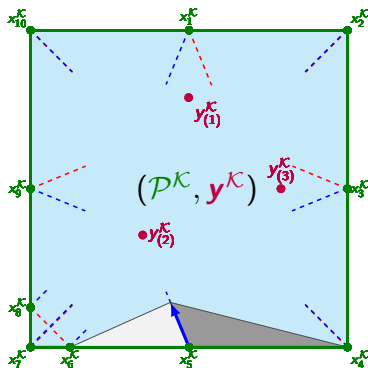
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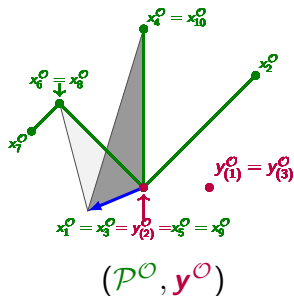
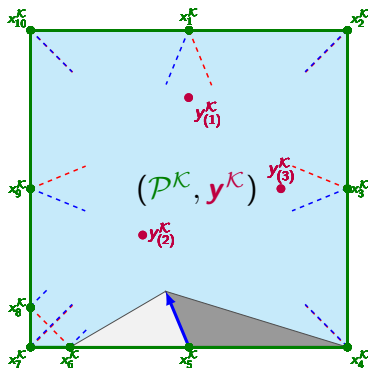
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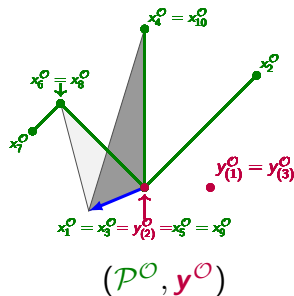
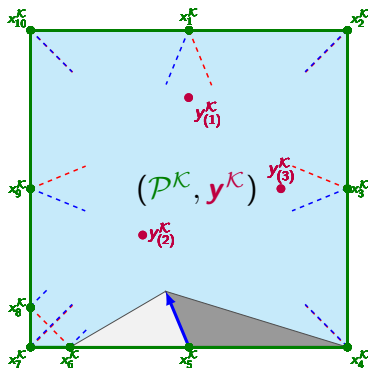
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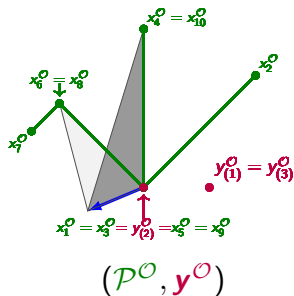
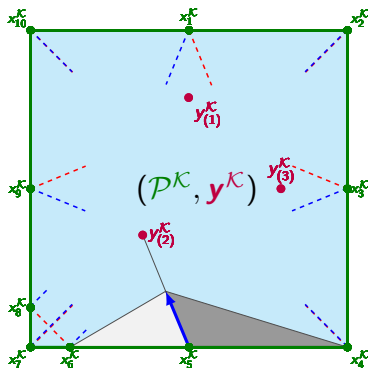
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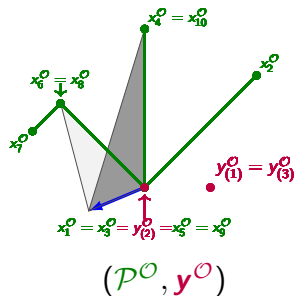
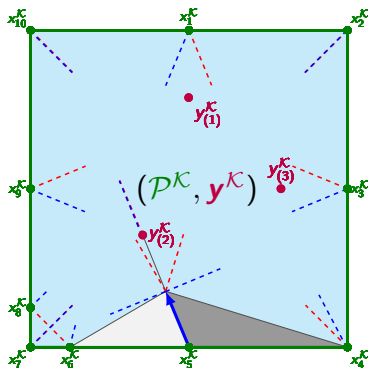
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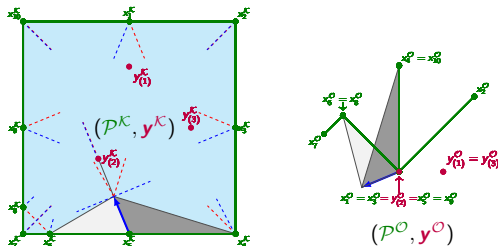
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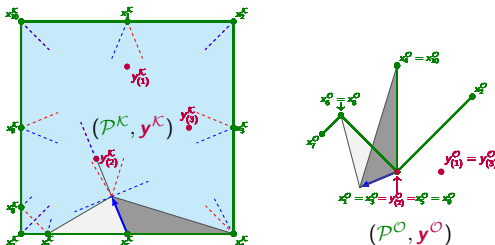
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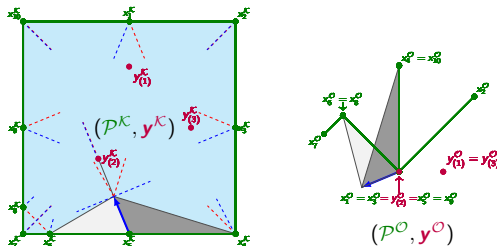
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Theorem (G.'25+)

The Origami Reconstruction Algorithm *works*.



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For all L -punctured polygons $(\mathcal{P}, \mathbf{y})$, it outputs a **valid, embedded** OCP.

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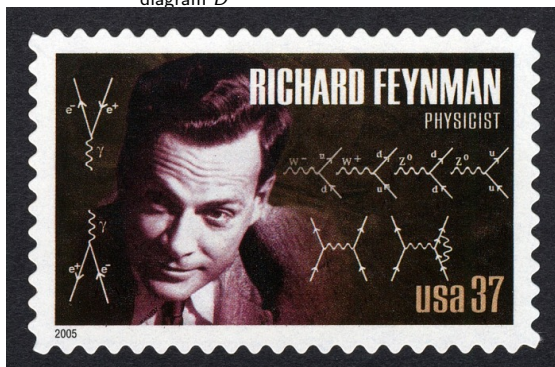
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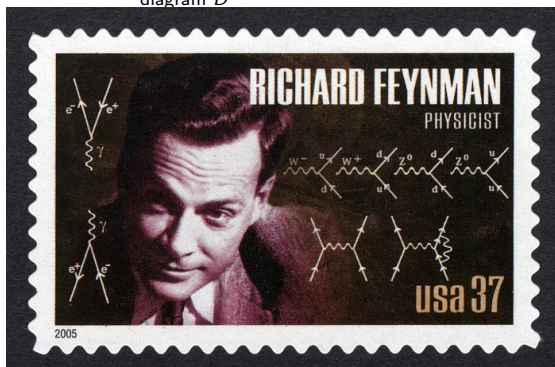
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Scattering amplitudes

Input: n particles with incoming momenta $P_1, P_2, \dots, P_n \in \mathbb{R}^{2,2}$ satisfying

- momentum conservation: $P_1 + P_2 + \dots + P_n = 0$;
- light-like: $P_i^2 = 0$.

Output: scattering amplitude $A(P_1, P_2, \dots, P_n)$.

Feynman diagram formalism (1948):

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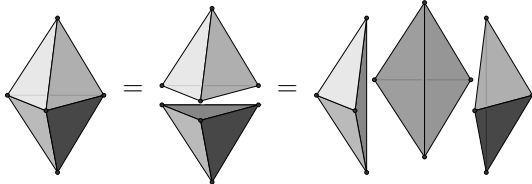
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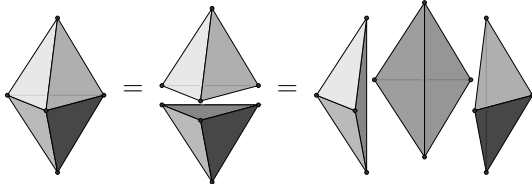


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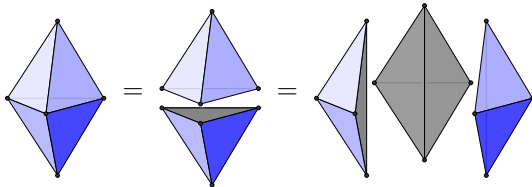


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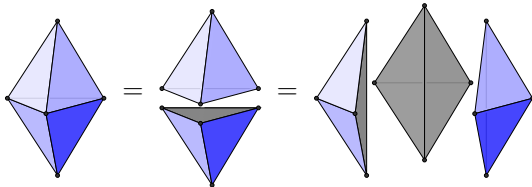


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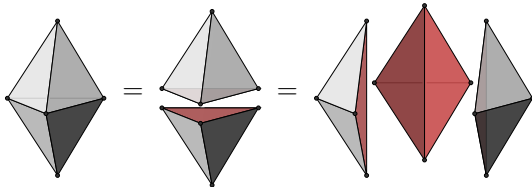


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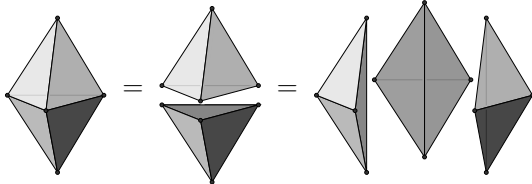


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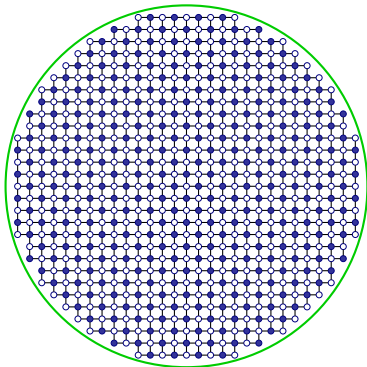
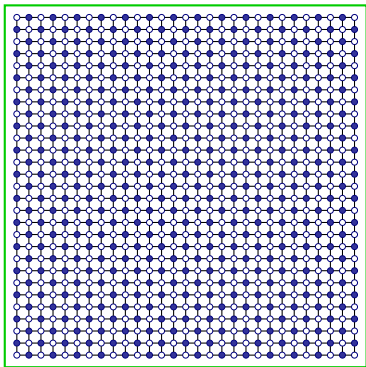
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- Our approach also implies that (2) is equivalent to (4). □

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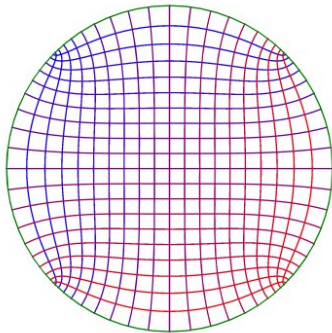
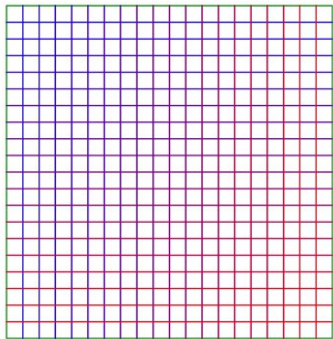
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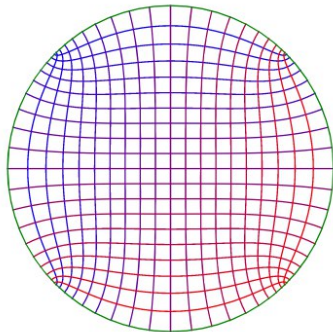
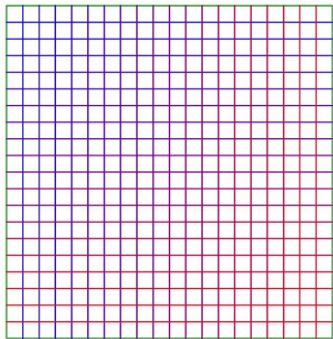
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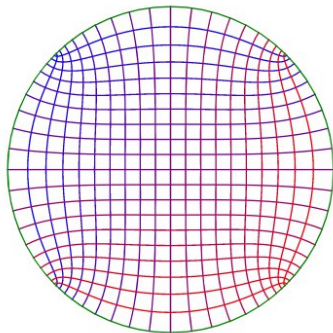
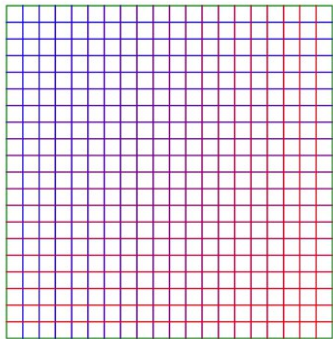
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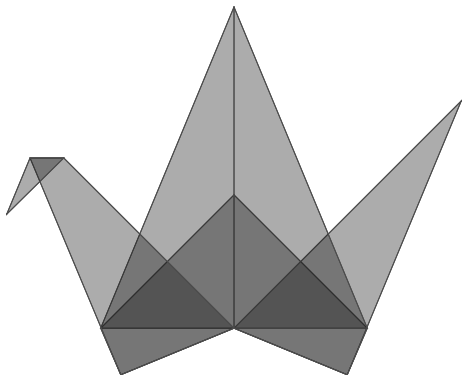
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Conjecture (OCP Existence Conjecture [KLRR '18], [CLR '21])

Every weighted planar bipartite graph Γ can be realized geometrically as a planar dual of an embedded OCP.



Thanks!